❖ *PAPERS ON* ❖

General Topology and Applications

TENTH SUMMER CONFERENCE AT AMSTERDAM

❖

Editors

Eva Coplakova
Klaas Pieter Hart

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Edited by Eva Coplakova and Klaas Pieter Hart

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PAPERS ON GENERAL TOPOLOGY AND APPLICATIONS[a]

TENTH SUMMER CONFERENCE AT AMSTERDAM

Editors
EVA COPLAKOVA AND KLAAS PIETER HART

Conference Organizers
J.M. AARTS, E. COPLAKOVA, F. VAN ENGELEN, K.P. HART, M.A. MAURICE, J. VAN MILL, AND M. TITAWANO

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CONTENTS

[a]The papers in this volume were presented at the Tenth Summer Conference on General Topology and Applications held in Amsterdam, the Netherlands on August 15–18, 1994.

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- BARUCH COLLEGE—CUNY
- THE CITY COLLEGE—CUNY
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- LONG ISLAND UNIVERSITY—C.W. POST CAMPUS
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- QUEENS COLLEGE—CUNY
- UNIVERSITY OF AUTÓNOMA METROPOLITANA, MEXICO CITY

Preface

J.M. AARTS, E. COPLAKOVA, F. VAN ENGELEN, K.P. HART, M.A. MAURICE, J. VAN MILL, AND M. TITAWANO

Department of Pure Mathematics
TU Delft
2628 CD Delft, the Netherlands

The Tenth Summer Conference on General Topology and Applications was held August 15–18, 1994 at Vrije Universiteit, Amsterdam. There were four special sessions at the conference: Continuum Theory and Dynamics, organized by Jan Aarts; Topology and Descriptive Set Theory, organized by Fons van Engelen; Set-Theoretic Topology, organized by Klaas Pieter Hart; and Infinite-dimensional and Geometric Topology, organized by Jan van Mill.

In addition there were two minicourses: Topological Methods in Surface Dynamics by Ph.L. Boyland and Topology and Descriptive Set Theory by A. Kechris.

The conference had over 180 participants with more than 100 contributed talks at the general and special sessions. There were, in addition, 20 invited talks. Plenary lectures were given by:

- J. W. Milnor on "Local Connectivity in Holomorphic Dynamics,"
- M. E. Rudin on "A Few Old Problems, Solved and Unsolved," and
- I. Moerdijk on "Groupoids, Local Equivalence Relations, and Monodromy."

The principal speakers and the topics at the special sessions were:

- F. Takens on "Topological Conjugacies, Moduli and Time Series,"
- A. W. Miller on "Descriptive Set Theory and Forcing,"
- B. Balcar on "Topologies on Complete Boolean Algebras," and
- R. Pol on "On Some Problems Concerning Weakly Infinite-dimensional spaces.

The organizers are grateful to Vrije Universiteit for hosting the conference and to the New York Academy of Sciences for publishing the proceedings. We also acknowledge and are grateful for the support (financial and otherwise) we received from: Technische Universiteit Delft, Vrije Universiteit Amsterdam, Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, Thomas Stieltjes Instituut, Stichting Mathematisch Centrum, Koninklijke Nederlandse Akademie van Wetenschappen, Nederlandse Organisatie voor Wetenschappelijk Onderzoek, European Research Office of the U.S. Army, Office of Naval Research European Office, Vereniging Trustfonds Erasmus Universiteit Rotterdam, Elsevier Science, Kluwer Academic Publishers, AKZO nv, IBM Nederland N.V., and Rabobank Nederland.

Finally, our sincere thanks to the referees, without whose diligence and timeliness it would not have been possible to publish this volume.

Dedication

On March 10, 1996 our friend and fellow organizer Maarten Maurice died. We dedicate these proceedings to his memory.

Jan Aarts
Eva Coplakova
Fons van Engelen
Klaas Pieter Hart
Jan van Mill
Marijke Titawano

Electronic Access for Topology Resources

B. BRECHNER,[a] M. MISLOVE,[b] D. SHAKHMATOV,[c] AND S. WATSON[d]

[a]*Department of Mathematics*
University of Florida
Gainesville, Florida 32611-8105

[b]*Department of Mathematics*
Tulane University
New Orleans, Louisiana 70118

[c]*Department of Mathematics*
Ehime University
Matsuyama, Japan
and
[d]*Department of Mathematics*
York University
North York, Ontario M3J 1P3 Canada

INTRODUCTION

Electronic communication is no longer the wave of the future, but a fact of today's life. E-mail now is almost universal for universities and colleges, and for businesses as well. For example, at many institutions — the University of Florida among them — almost all departmental business is conducted by e-mail. E-mail has the advantage of allowing us to communicate with our colleagues all over the world virtually instantaneously. Several electronic journals — journals that "publish" their issues electronically on the World Wide Web (WWW) — now are in operation. There has been a proliferation of anonymous ftp sites, including sites for mathematical papers, some of which (sites) focus on topology. A number of societies, including the American Mathematical Society, have web sites, and it now is possible to access the Mathematical Reviews on-line. The four of us strongly believe that it is important for the mathematical community to take advantage of, and participate in, the opportunities that the World Wide Web offers. At a personal level, these opportunities can make life easier by providing quick access to increasing amounts of information of all kinds, and by allowing instantaneous communication with colleagues. At the professional level, advances in information technology provide opportunities never before possible for increasing our research productivity and our awareness of new results, new ideas, conferences, and other events of interest to the mathematical community. The authors believe it is absolutely essential that we develop ways to take advantage of the internet so that the mathematical community is able to move into the twenty-first century as a full partner in the Information Age. The Information Age can be characterized as providing the possibility for people all over the world to know what is happening and to be able to communicate with one another in a timely fashion. We believe we must find effective uses for the tools this places at our disposal, as well as those which shortly will appear.

In that spirit, each of us is now involved in some kind of electronic project related to topology, and we wish to describe these projects below, because we feel they will be of interest to the entire topological community. The first author has started an electronic project called "Topology Eprints" at the University of Florida. This is basically an anonymous ftp/web site which is a repository for papers and abstracts in all areas of topology. There also is a newsletter associated with Topology Eprints. The second author has founded the electronic series, "Electronic Notes in Theoretical Computer Science" whose goal is the rapid electronic publication of conference proceedings, lecture notes and topical monographs using the World Wide Web. Finally, the last two authors are jointly developing a web site called "Topology Atlas." This web site is attempting to be all-inclusive. It includes abstracts and preprints of papers submitted to the site, as well as abstracts of papers on the Topology Eprints ftp site, a list of topology conferences all over the world, a list of "topology centers," open questions, publisher information, and much, much more. In the paragraphs below, we describe the purpose and vision we have for these activities.

TOPOLOGY EPRINTS

Beverly L. Brechner: ***brechner@math.ufl.edu***

The Department of Mathematics at the University of Florida has made a commitment to provide disk storage space for maintaining the archive of Topology Eprints. The archive is accessible through anonymous ftp:

ftp.math.ufl.edu/pub/topology

and is also accessible through a world wide web page:

http://www.math.ufl.edu/~teprints

E-mail to be read and answered manually, may be sent to:

teprints@math.ufl.edu

The Topology Eprints site is basically an ftp and web site, which is a repository for papers and abstracts in all areas of topology. Anyone can access and download papers by anonymous ftp on the Internet. If you have access to a web browser such as Netscape or Mosaic, you can access the web site, as well.

There is also an electronic newsletter associated with Topology Eprints. The newsletters contain information about conferences, open questions, and other items of interest. Melvin Henriksen is our Newsletter Editor.

Subscribers to our mailing list automatically receive both the newsletters and also the abstracts and announcements of new papers as they arrive. Thus, users can decide which papers will be of interest.

To add your name to our mailing list, write to: *teprints@math.ufl.edu* with the word *subscribe* on the subject line. Upon doing this, you will receive an e-mail message describing how to receive the Frequently Asked Questions (FAQ) document. The document will describe how to use the system via ftp; that is, how to download and upload papers and abstracts by ftp.

The Topology Eprints world wide web site is linked to its ftp site and to many other sites of interest to the topological community. In particular, there are links to the other two web sites discussed in this article. Anything from our web site may be printed directly on your own system if you are using a Unix system, and possibly also using other operating systems. If you cannot print directly from the web site, or cannot download from the web site, you will need to retrieve the information by ftp.

The first author wishes to express her appreciation to the many people who helped her get this project off the ground. These include Bob Flagg, Ralph Kopperman, and William Mitchell. And special thanks to University of Florida graduate student Scott Chastain, who has been acting as moderator/web developer of the system from the start, and who actually made it work! Without Scott's help, it wouldn't exist. And last, but not least, many thanks to the Department of Mathematics of the University of Florida for its support of this project.

ELECTRONIC NOTES IN THEORETICAL COMPUTER SCIENCE

Michael W. Mislove: ***mwm@math.tulane.edu***

Electronic Notes in Theoretical Computer Science (ENTCS) is meant to provide rapid, electronic publication of conference proceedings, lecture notes, and topical monographs. ENTCS is published electronically through the facilities of Elsevier Science B.V., and is affiliated with the journal Theoretical Computer Science. ENTCS is available at the URL

http://www.elsevier.nl/locate/entcs

All web users are allowed access to the Table of Contents and Abstracts of each volume of ENTCS, which also are published in TCS. Access to complete papers in the volumes is available on the web to those whose home institution maintains a subscription to TCS. It is anticipated that this arrangement will continue until at least the end of 1996. The Managing Editors of ENTCS are Michael Mislove (Tulane), Maurice Nivat (University of Paris), and Christos Papadimitriou (UC San Diego).

Topologists should find ENTCS an important resource, since it has a great deal of topology and related material in it. For example, Volume 1

http://www.elsevier.nl/locate/entcs/volume1.html

which is the Proceedings of the Eleventh MFPS meeting held last spring includes a number of papers on topology. These papers focus on the non-Hausdorff topologies and their use in programming semantics. Four of the thirty-one papers in Volume 1 are devoted to topology, and this area has had consistently high representation on the MFPS programs. Likewise, other theoretical computer science meetings often have topology papers in their programs; theoretical computer science has grown to be one of the most active areas of application for topology, using topology to solve the problems that arise there, and often providing the impetus for new research in topology.

The need for more rapid publication and dissemination of conference proceedings is due to the nature of most conferences in theoretical computer science. Such conferences usually involve a Call for Papers in response to which researchers submit papers for presentation at the meeting. The Program Committee then selects some of the submissions for presentation at the meeting, and the conference program is comprised of these papers perhaps along with some invited addresses. The process of requesting papers and then selecting them precedes the meeting by six months or more, so the papers already are somewhat dated when they are presented. Moreover, it is a common practice of conferences to distribute hard copy of their proceedings to the participants at the meeting. ENTCS is trying to eliminate further delays in the publication of this material by publishing the proceedings concurrently with the meeting itself. In addition, by utilizing the World Wide Web the material is disseminated much more broadly than can be achieved with the print media. We also allow conferences to distribute hard copy versions of their proceedings at their meeting if they wish, although we require that they have the same content and format as the electronic version.

Since conference proceedings of the type just described often consist largely of "extended abstracts" rather than complete journal papers, the editors expect and encourage conferences that publish their proceedings as volumes in ENTCS also to seek publication of journal versions of some of the papers in their proceedings with a major journal. For example, Volume 1 of ENTCS consists of the Proceedings of the Eleventh Conference on the Mathematical Foundations of Programming Semantics, held at Tulane University in March, 1995. A journal proceedings of this conference currently is in preparation. It will consist of expansions of some of the papers from the meeting, which have been written to journal standards. These submissions will be subject to the usual refereeing process, and when this process is complete, the journal version of the proceedings will appear as a special issue of Theoretical Computer Science.

While the primary motivation for founding ENTCS was to provide more rapid publication of conference proceedings as just described, there are other materials that would benefit from such publication. Among these are lecture notes and accompanying material for courses, as well as topical monographs of a timely nature. The series is published in volumes, each of which comprises a conference proceedings, a set of lecture notes, or a topical monograph. Publication of the material consists of placing the material in the ENTCS archive, where it is accessible from the web. This is accompanied by publication of the Table of Contents and the Abstracts of the papers in the volume in Theoretical Computer Science.

Because the editors want ENTCS to have the same appearance as high-quality print media, we have adopted some LaTeX macros that are used to prepare papers for publication in volumes in the series. The files for full papers are in PostScript, which has emerged as a universally accepted format for viewing and printing technical papers. The Table of Contents and Abstracts for each volume are prepared in HTML so that they can be viewed by any of the standard Web browsers.

One of the motivations for seeking an established publisher to help publish ENTCS was the editors' desire for a reliable archiving arrangement for ENTCS.

Elsevier will maintain the archive, and also will provide CD-ROM disks with accumulated volumes as appropriate. These CD-ROM disks will be distributed as part of subscriptions to TCS. The materials in ENTCS volumes are copyrighted by Elsevier. ENTCS volumes also will be reviewed by the usual review journals, such as Mathematical Reviews and Zentralblatt.

Conference organizers who are interested in publishing the proceedings of their meeting in ENTCS should contact one of the editors concerning their submission, or should send e-mail to me at *mwm@math.tulane.edu*. The list of editors is available at the ENTCS WWW site, as are details of what is required for a submission to ENTCS. Likewise, anyone having lecture notes or a topical monograph that he or she wishes to publish in ENTCS should contact one of the editors.

TOPOLOGY ATLAS

Dmitri Shakhmatov: *dmitri@ehimegw.dpc.ehime-u.ac.jp*
Stephen Watson: *stephen.watson@mathstat.yorku.ca*

Topology Atlas is a multi-purpose center for electronic distribution of information related to topology, a comprehensive attempt to create a "global village" in topology by taking advantage of recent advances in computer and Internet technology. Topology Atlas is designed to be a kind of "one-stop information shopping center" for those mathematicians and those members of the general public with some interest in topology. The publishers will try to accommodate and include any topic or any type of information related to topology which seems to be of some interest. Topology Atlas intends to be a complete historical and living portrait of the entire topological community, its endeavors (past and present), and its accomplishments — basically a living encyclopedia! Its purposes are two-fold. The first is to make the mathematical and scientific community aware of what topologists are doing. The second is to bring together topologists in the world community, establish communication, and promote joint work in a "living," interactive environment. We hope that this site will be a model for other areas of mathematics to emulate.

Topology Atlas is devoted to topology in the **broadest sense** possible. This includes (but is not limited to) low-dimensional topology, the topology of manifolds, knot theory, algebraic topology, differential topology, piecewise-linear topology, general topology, set-theoretic topology, geometric topology, continuum theory, plane topology, topological graph theory, topological algebra (groups, rings, fields and modules), topological vector spaces, topological aspects of functional analysis and C^*-algebras, topological questions of convex analysis and optimization, topological fixed point theory, descriptive set theory, topological problems in real and complex analysis, potential theory and partial differential equations, topological measure theory, convergence of measures and capacities, topology in computer science, digital topology and pattern recognition, topology in game theory, mathematical programming and mathematical economics. This list could be easily extended, but this gives a general direction.

As we are writing this (January 30, 1996), Topology Atlas is less than 2 months old, and the baby is growing very fast, so by the time this article appears in print in the "traditional" hard copy media, our current description will be hopelessly obsolete. Therefore, we decided to describe both current features of Topology Atlas, as well as planned future additions to them.

First we mention current section headings of Topology Atlas.

SECTION: *Who's Who in Worldwide Topology* collects lists of topologists who are currently "on-line." To be "on-line" means to have your home page stored in some computer in the world which can be accessed via one of the popular Internet protocols (hypertext transmission protocol [http], file transmission protocol [ftp], or gopher protocol). Your home page itself is nothing but a text file in the so-called HTML (hyper text markup language) format which contains some useful information about you, such as your mailing address, e-mail address, complete list of publications or just the most recent ones, perhaps a list of your graduate students, a list of your scientific interests, and any other information which you want to be widely known and available (all items mentioned above are optional). Once created and stored on some computer, your home page becomes a powerful vehicle for establishing new scientific contacts, because people who read Topology Atlas will be able to find information which you have placed in your home page within seconds. If you already have a homepage, please send its URL to Topology Atlas and it will be listed there. If you don't have a homepage but would like to have one, the publishers will be pleased to create a homepage for you and store it in our computer — all you need is just to send us the text you would like to have on your homepage, with or without links. We can also create and store homepages of "topological societies," i.e., research groups in topology united either by common research interests or geographical location, and homepages of topology "research centers" which usually contain lists of people in the same regional topology group.

SECTION: *Topology Happenings Around the World* lists conferences which are related to topology in our broad definition, topological seminars, and visiting topologists in various regions, as well as miscellaneous topological news and announcements.

SECTION: *Research in Topology* is one of the biggest in Topology Atlas. First, it has a comprehensive list of research topics in topology. Each contribution to this subsection is a short description of a particular topic written by a leading expert in the area so that browsers from the topological, mathematical, or scientific communities who are not specialists in that particular topic can become just a little bit familiar with the kinds of things that topologists work on. Following it is a collection of preprints, abstracts, research announcements, survey articles, and book descriptions, classified according to the special Topology Atlas Subject Classification of Topology. An important feature of this section is the collection of open problems in topology, which are subsequently discussed in the discussion forum.

SECTION: A recent addition to Topology Atlas, *Topological Commentary*, edited by Melvin Henriksen, is a newsletter devoted to publicizing items of a more personal nature to readers of Topology Atlas. TopCom is concerned with announcements about individuals or groups, and expressions of opinions on controversial matters. Exactly what will appear will depend on contributors and the interests of readers, but it will include: (1) Topological People — obituaries, interviews with major figures, and longer articles about topologists; (2) Editors and Topology — experiences involving attempts to publish papers in "general" journals where editors refuse to send them to a referee; (3) Personal Opinion — which may take the form of a letter to the editor or a column on some topic e.g., "Where is (some part of) topology going" or "What every young topologist should know" or "The effect of the current job market on topology;" (4) Historical reminiscences.

SECTION: *Publishing Topology* lists major publishers of topology, as well as journals which are known to publish topology at a reasonable scale.

SECTION: *Employment Opportunities* lists hirings in topology and publishes CV's of topologists looking for a job. Many more section headings are in the pipeline, so please stay tuned. Among planned future additions are a database of counterexamples in topology and a database of journals publishing topology, author's profiles, history of topology (with a collection of binary picture files reproducing rare and historic documents related to topology), collections of abstracts of talks presented at topological conferences, and many, many more, ...

The best way to read Topology Atlas is to use its interactive WWW (world wide web) site with URL (universal resource locator) address

http://www.unipissing.ca/topology/

The Web site is complemented by a variety of specialized mailing lists available through Topology Atlas Listserver. In the near future it is planned to have full ftp, telnet, and e-mail access to Topology Atlas, as well.

Topology Atlas welcomes and encourages submission of new information related to topology. The rules are simple: it suffices to send your information as an e-mail message to one of the publishers. We prefer ordinary (ASCII) text and are equally happy with LaTeX, AMS-TeX, plain TeX, and HTML but *any* format of electronic information is fine. Just send us your information as a regular e-mail message or, if you prefer, we are equally happy with attachments, floppy disks, anonymous ftp addresses, or URL links.

The publishers are proud to have the brilliant Topology Atlas team which makes Topology Atlas what it is. By the end of January 1996 this team has undergone extensive formation and development, but already includes many people whose expertise is a tremendous asset for Topology Atlas. As was already mentioned before, Melvin Henriksen is the Editor of TopCom. Klaas Pieter Hart is the Preprint Editor and John Schommer is the Editor for Conference Abstracts. Beverly Brechner is the Editor of the Continuum Theory section and Murat Tuncali is the Local Editor at Nipissing University. Kim Price is the Art Director. Additional appointments are forthcoming. The publishers are especially pleased to acknowledge World Wide Web coöperation and assistance of Nipissing University (North Bay, Ontario, Canada), the host of the 12th Summer Conference on Topology and its Applications

and Workshop on Continuum Theory, Set-Theoretic Topology, and Applied Topology (August 12–16, 1997). We are grateful for the support of Murray Green (Vice-President, Administration and Finance), Ted Chase (Dean of Arts and Science), and Robert Bergquist (Associate Dean of Arts and Science). Our special thanks go to the Technical Support Team from the Department of Computing and Communication Services: Frank Ciancio (Internet Specialist), Andre Roy (Programmer/Analyst), and Greg Seamen (Technical Support Technologist).

CONCLUSION

And so, dear colleagues, we hope we have given you enough of a portrait of our electronic endeavors to whet your appetite. In particular, we hope you take advantage of these and all the other electronic opportunities that are now available to you. But we note that, in order for these electronic ventures to be successful, it is necessary that you actively participate by contributing to these endeavors yourselves. One must "give" in order to "receive."

Let us look to the future! For the future is NOW!

NOTE ADDED IN PROOF: Topology Eprints has now merged with Topology Atlas. The University of Florida ftp site has become a mirror for the new Topology Atlas ftp site. To submit information to Topology Atlas, or ask questions, send e-mail to:

atlas@yorku.ca

To retrieve articles by ftp from the University of Florida, use the eddress:

topology-atlas-eprints@math.ufl.edu

A Note on Holsztyński's Theorem

J. ARAUJO[a,d], J. J. FONT[b,e], AND S. HERNÁNDEZ[b,f]

[a]*Departamento de Matemáticas, Estadística y Computación*
Facultad de Ciencias
Universidad de Cantabria
Avenida de los Castros s.n., E-39071
Santander, Spain

[b,c]*Departamento de Matemáticas*
Universitat Jaume I
Campus Penyeta Roja, E-12071
Castellón, Spain

ABSTRACT: In this note we provide an elementary proof of the following generalization of Holsztyński's theorem [2]: If there exists a linear isometry T of a completely regular subspace A of $C_0(X)$ into $C_0(Y)$, then there is a subset Y_0 of Y and a continuous map h of Y_0 onto X and a continuous map $a: Y_0 \to \mathbf{K}$, $|a| \equiv 1$, such that $(Tf)(y) = a(y)f(h(y))$ for all $y \in Y$ and all $f \in A$.

As a consequence, we extend to $C_0(X)$-spaces an old result by Myers [3].

INTRODUCTION

Let $\mathbf{K}$ denote the field of real or complex numbers. For a locally compact Hausdorff space X, we denote by $C_0(X)$ the Banach space of all continuous $\mathbf{K}$-valued functions defined on X which vanish at infinity, equipped with its usual supremum norm. If X is compact, we write $C(X)$ instead of $C_0(X)$. $X \cup \{\infty\}$ denotes the Alexandroff compactification of X.

The well-known Banach-Stone theorem states that if there exists a linear isometry T of $C_0(X)$ onto $C_0(Y)$, then there is a homeomorphism h of Y onto X and a continuous map $a: Y \to \mathbf{K}$, $|a| \equiv 1$, such that

$$(Tf)(y) = a(y)f(h(y)) \text{ for all } y \in Y \text{ and all } f \in C_0(X).$$

A generalization of the above theorem was given by Holsztyński [2]. He proved that if there exists a linear isometry T of $C(X)$ into $C(Y)$, then there is a closed subset Y_0 of Y and a continuous map h of Y_0 onto X and a continuous map $a: Y_0 \to \mathbf{K}$, $|a| \equiv 1$, such that

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[d]Research partially supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT, PS90-100). E-mail address: araujoj@ccaix3.unican.es.
[e]E-mail address: font@mat.uji.es.
[f]Research partially supported by Fundació Caixa Castelló (A-39-MA). E-mail address: hernande@mat.uji.es.

$$(Tf)(y) = a(y)f(h(y)) \text{ for all } y \in Y_0 \text{ and all } f \in C(X).$$

Let us recall that a linear subspace A of $C_0(X)$ is said to be completely regular if for every $x \in X$ and every open neighborhood U of x, there exists $f \in A$ with $\|f\| = |f(x)| = 1$ and $|f| < 1$ outside U.

In this note we provide a proof of an extension of Holsztyński's theorem for $C_0(X)$-spaces; namely, completely regular subspaces. Our proof does not depend on extreme points , T-sets, M ideals, ... (see [1]), but on elementary concepts instead. Furthermore, despite the lack of constant functions, the isometry can be written as a weighted composition map.

As corollary we extend to $C_0(X)$-spaces the following result due to Myers [3]: Let **K** be the real numbers. Then a sufficient condition for the compact spaces X and Y to be homeomorphic is that a completely regular linear subspace of $C(X)$ and such a subspace of $C(Y)$ be isometrically isomorphic.

THEOREM: If there exists a linear isometry T of a completely regular subspace A of $C_0(X)$ into $C_0(Y)$, then there is a subset Y_0 of Y and a continuous map h of Y_0 onto X and a continuous map $a: Y_0 \to \mathbf{K}$, $|a| \equiv 1$, such that

$$(Tf)(y) = a(y)f(h(y)) \text{ for all } y \in Y_0 \text{ and all } f \in A.$$

Proof: For all $x \in X$, let $C_x = \{f \in A: 1 = \|f\| = |f(x)|\}$. For any $f \in A$, let $L(f) = \{y \in Y: \|Tf\| = |(Tf)(y)|\}$ and let $I_x = \bigcap_{f \in C_x} L(f)$. To prove that I_x is nonempty for all $x \in X$ and, since I_x is a closed subset of $M_f = \{y \in Y: |(Tf)(y)| \geq \|Tf\|/2\}$, which is compact for any $f \in C_x \subset C_0(X)$, it suffices to prove that if $f_1, \ldots, f_n$ belong to C_x, then $\bigcap_{i=1}^{n} L(f_i) \neq \varnothing$. We have that $1 = \|f_i\| = |f_i(x)|$ for all $i = 1, \ldots, n$. Let $f \in A$ defined as $\sum_{i=1}^{n} (|f_i(x)|/f_i(x))f_i$. Clearly $|f(x)| = n = \|f\|$. Since T is an isometry, $\|Tf\| = n$ and there is $y \in Y$ such that $n = |(Tf)(y)| = \sum_{i=1}^{n} |(|f_i(x)|/f_i(x))(Tf_i)(y)|$. As $\|Tf_i\| \leq 1$ for all $i = 1, \ldots, n$, we deduce that $|(Tf_i)(y)| = 1$ for all $i = 1, \ldots, n$, that is, $y \in \bigcap_{i=1}^{n} L(f_i)$.

We next show that, given $x_0 \in X$, if $f \in A$ and $f(x_0) = 0$, then $Tf(y) = 0$ for all $y \in I_{x_0}$. If there exists $y_0 \in I_{x_0}$ such that $Tf(y_0) \neq 0$ for some $f \in A$, we can assume that $\|f\| = 1$ and $Tf(y_0) = \alpha$ with $0 < \alpha \leq 1$. Let $U = \{x \in X: |f(x)| \geq \alpha/2\}$. Since A is completely regular, there is $g \in A$ such that $1 = \|g\| = |g(x_0)|$, $|g(x)| < 1$ for all $x \in U$ and, multiplying by a constant if necessary, $Tg(y_0) = 1$. Since U is compact, we can consider $s := \sup_{x \in U}\{|g(x)|\} < 1$. Then there exists $M > 0$ such that $1 + Ms < \alpha + M$. Take $x \in U$. Then $|(f + Mg)(x)| \leq 1 + Ms$. If $x \notin U$, then $|(f + Mg)(x)| \leq \alpha/2 + M$. Hence $\|f + Mg\| < \alpha + M$, but $\alpha + M = (Tf)(y_0) + (MTg)(y_0) \leq \|T(f + Mg)\|$, which is a contradiction.

Let us now show that $|f(x_0)| = |Tf(y)|$ for all $y \in I_{x_0}$ and all $f \in A$. It is enough to check that if $|f(x_0)| = 1$ for some $f \in A$, then $|Tf(y)| = 1$ for all $y \in I_{x_0}$. Indeed there exists $g \in A$ such that $1 = \|g\| = |g(x_0)|$. From the definition of I_{x_0}, we know that $|Tg(y)| = 1$ for all $y \in I_{x_0}$. Multiplying by a constant if necessary, we will assume that $f(x_0) = g(x_0)$. Hence, $(f - g)(x_0) = 0$ and, by the preceding paragraph, $|Tf(y)| = |Tg(y)| = 1$ for all $y \in I_{x_0}$.

To prove that $I_x \cap I_{x'} = \varnothing$ whenever x, x' are distinct elements of X, take $f \in A$ such that $|f(x)| \neq |f(x')|$. By applying the above arguments, it is easy to verify that if $y \in I_x$ and $y' \in I_{x'}$, then $|(Tf)(y)| = |f(x)| \neq |f(x')| = |(Tf)(y')|$. So $I_x \cap I_{x'} = \varnothing$.

Let Y_0 be the set $\bigcup_{x \in X} I_x$ and let h be a map defined of Y_0 onto X by the requirement that $h(y) = x$ if $y \in I_x$. This map is well-defined since the elements of the family $\{I_x: x \in X\}$ are pairwise disjoint and it is onto from the fact that $I_x \neq \emptyset$ for every $x \in X$.

To prove the continuity of h, suppose that $h(y_0) = x_0$ for some $y_0 \in Y_0$. Let U be an open neighborhood of $x_0 \in X$ and , since A is completely regular, let $f \in A$ such that $f(x_0) = \|f\| = 1$ and $|f(x)| < 1$ for all x outside U. Let $s = \sup_{x \in X-U} |f(x)| = \sup_{x \in (X \cup \{\infty\})-U} |f(x)|$. It is easy to see that $s < 1$. Since $y_0 \in I_{x_0}$, $|(Tf)(y_0)| = \|Tf\| = 1$. Let $V = \{y \in Y_0 : |(Tf)(y)| > s\}$. Clearly V is an open neighborhood of y_0. Let $y \in V$. Hence, $|f(h(y))| = |(Tf)(y)| > s$; that is, $h(y) \in U$.

Finally, let us define a map a of Y_0 into $\mathbf{K}$ as follows: given $y \in Y_0$, let f any function in A such that $f(h(y)) = 1$. Hence, we define $a(y) = (Tf)(y)$ for all $y \in Y_0$. This is a well-defined map because if we take another function g in A such that $g(h(y)) = 1$, then $(f-g)(h(y)) = 0$ and, as we have seen above, $(Tf)(y) = (Tg)(y)$. Moreover, $|a(y)| = 1$ for all $y \in Y_0$.

Next we obtain the multiplicative representation of T. We have already proved that if $f(h(y)) = 0$, then $(Tf)(y) = 0$ for all $y \in Y_0$ and all $f \in A$. If $f(h(y)) \neq 0$ for some $f \in A$ and some $y \in Y_0$, then let $g = f - f(h(y))k$, k being any function in A such that $k(h(y)) = 1$. Clearly $g(h(y)) = 0$. Thus, $(Tg)(y) = 0$; that is, $(Tf)(y) = a(y) f(h(y))$.

In order to prove the continuity of a, fix $y \in Y_0$ and consider any $f \in A$ such that $f(h(y)) \neq 0$ and let $W = \{x \in X: f(x) \neq 0\}$. It is clear that $h^{-1}(W)$ is an open neighborhood of y. Moreover, the map $Tf/(f \circ h)$ is continuous on $h^{-1}(W)$ and a and $Tf/(f \circ h)$ coincide on $h^{-1}(W)$. This completes the proof. ❑

REMARK: The following example shows that the subset Y_0 of Y may not be closed. Let T be the isometric embedding of $C_0(N)$ into $C(N \cup \{\infty\})$, where N are the positive integers. It is easy to check that $Y_0 = N$.

COROLLARY: If there exists a linear isometry T of a completely regular subspace A of $C_0(X)$ onto such a subspace B of $C_0(Y)$, then there is a homeomorphism h of Y onto X and a continuous map $a: Y \to \mathbf{K}$, $|a| \equiv 1$, such that

$$(Tf)(y) = a(y) f(h(y)) \text{ for all } y \in Y \text{ and all } f \in A.$$

Proof: Let us consider the inverse of T, T^{-1}, which is an isometry of B onto A. As in the proof of the theorem above, we obtain a subset X_0 of X and a continuous map k of X_0 onto Y. Given $y_0 \in Y$, there exists $x_0 \in X_0 \subset X$ such that $k(x_0) = y_0$. Hence, $|T^{-1}g(x_0)| = |g(y_0)|$ for all $g \in B$. Thus, $|T^{-1}(Tf)(x_0)| = |Tf(y_0)|$ for all $f \in A$, i.e., $y_0 \in I_{x_0}$. This implies both that $Y_0 = Y$ and that the inverse of h is the continuous map k, which is defined on $X_0 = X$. Summing up, h is a homeomorphism of Y onto X and the remainder of the proof follows from the theorem above. ❑

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The Gleason Cover of a Flow

RICHARD N. BALL[a] AND JAMES N. HAGLER[b]

Department of Mathematics and Computer Science
University of Denver
Denver, Colorado 80208

ABSTRACT: We prove the existence and uniqueness of projective covers in the category of flows with perfect flow maps. The main point is that Gleason's classical results on the existence and uniqueness of the projective cover of a compact Hausdorff space hold when all spaces are equipped with (a fixed monoid of continuous) actions and all maps are required to respect them.

1. INTRODUCTION

In his celebrated paper [21], Gleason not only showed the existence and uniqueness of projective covers in the category of compact Hausdorff spaces and continuous maps, he characterized them as precisely the extremally disconnected spaces in that category. The corresponding characterization in flows is much harder, and we take pains in this article to elucidate some of the subtleties involved by analyzing several examples. In fact, this paper is the first of several [6], [7], [8], [9] whose common objective is the aforementioned characterization. We must confess, however, that we have not altogether realized this objective at the time of this writing.

Before describing our results we need some definitions and notation. The category of Hausdorff topological spaces with continuous functions will be denoted **Sp**, while **K**, and **Tych** denote the full subcategories of compact and Tychonoff spaces, respectively. (Unless otherwise noted, we assume throughout that spaces are Hausdorff and that maps between them are continuous.) We reserve the letter T to denote a fixed topological monoid with identity, whose elements we term *actions*. We say that *T acts on a space X* if there is a monoid homomorphism $\phi_X : T \to \hom_{\mathbf{Sp}}(X, X)$. We suppress mention of ϕ_X, writing $\phi_X(t)$ as t_X or simply t, and denote the action of t on x by tx. In this notation, T acts on X if $1x = x$ for all $x \in X$, where 1 is the monoid identity, and $(t_1t_2)x = t_1(t_2x)$ for all $t_1, t_2 \in T$ and $x \in X$.

Two special actions deserve mention. We say that *T acts trivially* on X if $tx = x$ for all $t \in T$ and $x \in X$, i.e., if $\phi_X(t) = 1$ for all $t \in T$. And if $T = \{1\}$ is the trivial monoid then we are in the classical situation of no (nontrivial) actions, and our development reproduces the Gleason cover.

A *flow* is a pair (X, e), where X is a space on which T acts in such a way that the evaluation map $e : T \times X \to X$ (defined by $e(t, x) = tx$) is continuous. Usually the map e is clear from context, so we refer to the flow (X, e) as simply X. A *flow*

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[a]E-mail: rball@du.edu.
[b]E-mail: jhaglerl@du.edu.

morphism (or *flow map*) $f: Y \to X$ is a continuous function between flows which commutes with the actions, i.e., $f(ty) = tf(y)$ for all $y \in Y$ and $t \in T$. We denote the category of flows and flow maps by **TSp**, and we likewise define **TK** and **TTych**.

A subspace Z of a space Y on which T acts is *T-invariant* if $tz \in Z$ for all $z \in Z$ and $t \in T$. A T-invariant subspace of a flow is called a *subflow*. A map $f: Y \to X$ is *T-irreducible* if it is a surjective flow function which maps no proper closed subflow of Y onto X. Clearly T-irreducibility reduces to the usual notion of irreducibility in the case of no actions. If f and g are T-irreducible and if f is closed then gf is T-irreducible; conversely, if gf is T-irreducible and if f is onto, then f and g are T-irreducible. We say that a flow Y is a *(proper) T-irreducible preimage* of X if there is a (noninjective) T-irreducible flow map $f: Y \to X$.

Recall that a continuous function $f: Y \to X$ is *perfect* if it is a closed map such that $f^{-1}x$ is compact for each $x \in X$. If f and g are perfect maps then gf is perfect; conversely, if gf is perfect then f is perfect, and also if f is surjective, then g is perfect [22, Section 2]. When the morphisms are required to be perfect, the notations for the categories corresponding to those mentioned above are $\mathbf{TSp^P}$, and $\mathbf{TTych^P}$.

An object P in a category C is a *projective* if $P \in C$, and, if for every f and g in C, f surjective, there is some k in C which makes the diagram commute. If C is a category of flows then we say that P is the *projective cover* (or *absolute* or *T-*

$$\begin{tikzcd}
 & Y \arrow[d, "f"] \\
P \arrow[ur, "k"] \arrow[r, "g"'] & X
\end{tikzcd}$$

Gleason cover) of $X \in C$ if P is a projective in **C** and there is a perfect T-irreducible map $f: P \to X$. We denote the projective cover of the flow X by $\gamma^T X$.[1]

We can now describe our results and the organization of the remainder of this paper in more detail. When the action monoid T is actually a compact topological group and X is a compact Hausdorff flow, we provide in Section 2 a complete characterization of the projective cover of X. These results recapture Gleason's by taking $T = \{1\}$. In Theorems 2.6, 3.8, and 3.9, the main results of this paper, we prove the existence and uniqueness of the projective cover in the categories **TK** and $\mathbf{TTych^P}$.

Sections 4 and 5 may be read independently of Section 3 if the reader is willing to accept the existence and uniqueness of the projective cover. We show in a very brief Section 4 how the universal minimal flow of a topological semigroup can be constructed as a projective cover. In Section 5 we give a number of examples which illustrate both the scope of our results and the difficulty of providing a characterization of the projective cover of a flow in the same spirit as that of the no-action Gleason cover. We also provide Examples 5.1 and 5.4, which show that

[1]The notation $\gamma^T X$ is actually somewhat ambiguous, since the same monoid T can act on the same space X in different ways. But we trust that it can be used without confusion in the sequel.

the characterization of the projective cover when T is a compact group given in Section 2 is no longer valid when either the action monoid is not compact or not a group. We also show in Example 5.6 that the projective cover can have surprising and unexpected properties.

In Section 6 we provide an "algorithm" for constructing the projective cover for compact flows and for arbitrary monoids. This procedure is summarized and captured in what, for obvious reasons, we have named the Wretched Diagram. (See Subsection 6.3 and Theorem 6.6.) We believe that, in spite of its wretchedness, the diagram sheds new light on the structure of the projective cover, and we show how many of our earlier results (e.g., in the compact group case) and some new ones can be recovered from this general algorithm. For example, in Theorem 6.9 we derive the result of Balcar and Franek [3] that a universal minimal flow of a discrete semigroup is extremally disconnected. Finally, Section 7 contains a brief discussion of open problems.

2. A COMPACT GROUP OF ACTIONS

The most straightforward case is that in which the monoid of actions is a compact group. The importance of this case is twofold. First, we are able to characterize the projective objects in **TK** completely in terms of known objects, such as classical Gleason spaces and T itself, and second, this case points out the problems inherent in the general situation. *Accordingly, throughout this section we assume that T is a compact group.*

Suppose that $X \in \mathbf{TK}$. Define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if and only if $x_2 = tx_1$ for some $t \in T$. Let X/T denote the quotient space, and let $q : X \to X/T$ denote the quotient map.

LEMMA 2.1: If $X \in \mathbf{TK}$ then X/T is a compact Hausdorff space. Furthermore, when X/T is equipped with trivial action, the quotient map q becomes a T-irreducible flow surjection.

Proof: First note that each equivalence class $[x] = Tx = \{Tx : t \in T\}$ is compact. So consider $x \in X$ and open set U containing $[x]$. Put $K = X \backslash U$. Then TK is a closed subset of X and $TK \cap [x] = \varnothing$. Put $V = X \backslash TK$. Then $TV \subseteq V$ and $[x] \subseteq V \subseteq U$. Thus by Kelley [26, p. 148] X/T is Hausdorff. And it is clear that q is a T-irreducible flow surjection. ❑

We continue by examining the case in which T acts trivially on $Y \in \mathbf{TK}$. In this case we can consider $T \times Y$ to be in **TK** also by defining the action of $t \in T$ to be $t(t_1, y) = (tt_1, y)$. Then the projection map $p : T \times Y \to Y$, defined by $p(t, y) = y$, is a flow surjection which is T-irreducible because T is a group.

LEMMA 2.2: Suppose that T acts trivially on the compact flow Y. Then any flow surjection $f \in \mathbf{TK}$ "drops" to a unique flow surjection g which makes this diagram commute.Furthermore, f is T-irreducible if and only if g is T-irreducible.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, "h"] & T \times Y \arrow[d, "p"] \\ X/T \arrow[r, "g"'] & Y \end{tikzcd}$$

Proof: Let h denote pf, and consider $x \in X$, $f(x) = (t_1, y)$ and $t \in T$. Then

$$f(tx) = tf(x) = (tt_1, y),$$

so $h(tx) = y = h(x)$. This shows that g is well-defined. Now let $U \subseteq Y$ be open. Then $f^{-1}U$ is open if and only if $q^{-1}f^{-1}U$ is open if and only if $h^{-1}U$ is open, which it is. Thus f is continuous. If f is T-irreducible then, because p is T-irreducible and f is closed, h is T-irreducible. Since $h = gq$ and q is onto it follows that g is T-irreducible. Conversely, if g is T-irreducible then, because q is T-irreducible and closed, h is T-irreducible. Since $h = pf$ and f is onto, it follows that f is T-irreducible. ❑

PROPOSITION 2.3: Suppose T acts trivially on the extremally disconnected compact Hausdorff space Y. Then the flow $T \times Y$ has no proper T-irreducible pre-images.

Proof: Assuming the notation of Lemma 2.2, suppose that f is a T-irreducible surjection. Then g is also a T-irreducible surjection, and since the action is trivial on both X/T and Y, g is irreducible in the ordinary sense. Since Y is extremally disconnected it follows [34, p. 285] that g is a homeomorphism. What we must do is show that f is injective. To this end consider $x_i \in X$ such that $f(x_1) = f(x_2) = (t_1, y)$. Then $gq(x_1) = gq(x_2)$, and so $q(x_1) = q(x_2)$ since g is a homeomorphism. Hence there exists $t \in T$ such that $tx_1 = x_2$. But now

$$(t_1, y) = f(x_2) = f(tx_1) = tf(x_1) = t(t_1, y) = (tt_1, y),$$

so $t_1 = tt_1$. Since T is a group, $t = 1_T$, and so $x_1 = x_2$. ❑

Proposition 2.3 is one key to the T-Gleason cover of a compact flow. The other key is a result which says that any two T-irreducible maps onto the same compact flow have a common ancestor.

LEMMA 2.4: For every pair of T-irreducible maps f_i in **TK** with common codomain there exists a pair of T-irreducible maps g_i in **TK** with common domain which make the diagram commute.

$$\begin{tikzcd} Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

Proof: Let $W = \{(y_1, y_2) \in Y_1 \times Y_2 : f_1(y_1) = f_2(y_2)\}$. Then $W \in \mathbf{TK}$ if we define $t(y_1, y_2) = (ty_1, ty_2)$. Let $q_i : W \longrightarrow Y_i$ denote the projection onto Y_i for $i = 1, 2$, and let $f = f_1 q_1 = f_2 q_2$. Using Zorn's lemma, select a closed T-invariant subspace $Z \subseteq W$ minimal with respect to $f(Z) = X$. It is easily checked that we are done if we define $p_i = q_i|Z$. ❑

PROPOSITION 2.5: Suppose T acts trivially on the extremally disconnected compact Hausdorff space Y. Then $T \times Y$ is a projective in $\mathbf{TK}$.

Proof: Consider the test maps f and g. By Lemma 2.4 there are T-irreducible maps p_i in $\mathbf{TK}$ which make the diagram commute. But p_1 is a flow homeomor-

$$\begin{tikzcd} Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\ T \times Y \arrow[r, "g"'] \arrow[ur, "k"] & X \end{tikzcd}$$

phism by Proposition 2.3, hence the desired map is $k = p_2 p_1^{-1}$. ❑

We have now assembled the machinery necessary to state and prove the main result of this section. Let us denote the Gleason cover of the compact Hausdorff space Y by γY. For a given $X \in \mathbf{TK}$ let $q : X \rightarrow X/T = Y$ denote the quotient map. Regard both Y and γY as objects in the category $\mathbf{TK}$, with T acting trivially on both, and let P denote the product flow $T \times \gamma Y$, which is a projective in TK by Proposition 2.5. Let $\gamma : \gamma Y \rightarrow Y$ denote the Gleason map, and let $p : P \rightarrow \gamma Y$ be the projection map. The composition γp is T-irreducible since the factors are closed

$$\begin{tikzcd} P \arrow[r, "\gamma_X^T"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ \gamma Y \arrow[r, "\gamma"'] & Y \end{tikzcd}$$

and T-irreducible; let γ_X^T be a flow map produced by the fact that P is a projective. Since q is surjective and γ_X^T is closed, γ_X^T is onto, and hence T-irreducible. We have proved the main theorem of this section.

THEOREM 2.6: Suppose that T is a compact group acting on the compact flow X. Then the projective cover of X in $\mathbf{TK}$ is

$$\gamma^T X = T \times \gamma(X/T),$$

and γ_X^T is the canonical T-irreducible surjection onto X.

We give some simple corollaries of this result. Note that the first of these can be proved simply and directly.

COROLLARY 2.7: Let T be a compact group, acting on itself by left multiplication. Then $\gamma^T T = T$.

Proof: The space T/T has one element, hence is extremally disconnected. The result follows immediately. ❑

COROLLARY 2.8: Let T be a compact group, acting on itself by conjugation. Then $\gamma^T T = T \times \gamma T^c$, where T^c denotes the compact space of conjugacy classes of T, and the action is trivial on the right coordinate and left multiplication on the left coordinate. In particular, if T is Abelian then $\gamma^T T = T \times \gamma T$. If T is finite and Abelian, then $\gamma^T T = T \times T$.

COROLLARY 2.9: The projectives in **TK** are the flows of the form $T \times Y$, where Y is an extremally disconnected compact Hausdorff space, and where the action is left multiplication on the left coordinate and trivial on the right coordinate.

We can recast these results in terms of a natural ordering of the T-irreducible preimages of a single compact flow X as follows. Suppose $f_i : Y_i \to X$ are two T-irreducible maps in **TK**. Let us agree to say that $Y_1 \geq Y_2$ if there is a T-irreducible map $g : Y_1 \to Y_2$ such that $f_2 g = f_1$. This relation imposes a preorder on the class of T-irreducible preimages of X in **TK**.

COROLLARY 2.10: For every $X \in \mathbf{TK}$ there are $Y, Z \in \mathbf{TK}$ on which T acts trivially such that Z is extremally disconnected and $Y \leq X \leq T \times Z$. X and Z are unique up to homeomorphism with respect to these properties. Furthermore, $T \times Z$ is an upper bound of all the upper bounds of X, and Y is an upper bound of all the lower bounds of X on which T acts trivially.

We close this section with a question. What are the equivalence classes of this preordering? That is, what can one say of two T-irreducible preimages Y_1 and Y_2 of a compact flow X if $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$? In particular, are they flow homeomorphic? In the irreducibility ordering on compact Hausdorff spaces (with no actions), the map which witnesses the fact that $Y_1 \geq Y_2$ is unique [22], and all members of a preorder class are homeomorphic. It is easy to find examples in which the flow map which witnesses $Y_1 \geq Y_2$ is far from unique, yet we know of no instance in which members of a preorder class are not flow homeomorphic.

3. THE EXISTENCE AND UNIQUENESS OF THE PROJECTIVE COVER

In this section we prove the existence of the projective cover in the categories **TK** and **TTych^P**. The method of proof is to construct the projective cover of the flow X by first taking the pullback Z of the perfect T-irreducible preimages of X and then passing to a closed subflow Z_0 of Z minimal with respect to mapping onto X. This approach has the virtues of being ultimately direct and conceptually simple; it is due to Banashewski [10] and was used by Hager [23] to construct the classical Gleason cover of a compact Hausdorff space. The approach works because, though none of the categories of flows we discuss in this article are topological in the sense of Preuss [30], they do have several attractive properties which allow familiar constructions.

(a) All mentioned categories are co-well-powered. That is, for a given flow X there are (up to flow isomorphism) only a set's worth of perfect T-irreducible preimages of X. This is a consequence of the fact that there is a cardinality bound on such preimages (Proposition 3.6).
(b) All mentioned flow categories have pullbacks (Proposition 3.1).
(c) For every $\mathbf{TSp^P}$ surjection $f : Y \to X$ there is an embedding $g : Z \to Y$ such that fg is T-irreducible (Proposition 3.2). Note that Z need not be unique.

3.1. Pullbacks

Suppose we are given a set $\{f_i : i \in I\}$ of morphisms with common codomain in a category $\mathbf{C}$. Consider the class $\mathcal{P}$ of families $\{p_i' : i \in I\}$ of morphisms in $\mathbf{C}$ with common domain such that $f_i p_i' = f_j p_j'$ for all $i, j \in I$. Then the *pullback* of $\{f_i : i \in I\}$ is a member of $\mathcal{P}$ which is universal in the sense that for any $\{p_i' : i \in I\} \in \mathcal{P}$ there is a unique morphism p such that $p_i p = p_i'$ for all $i \in I$. We usually use f to denote $f_i p_i$, whose definition is independent of i, and by abusing the terminology we refer to the common domain of the p_i's as the pullback of the f_i's. The category $\mathbf{C}$ is said to *have pullbacks* if every set of morphisms with common codomain has a pullback. For example, in spaces the pullback of a set of continuous functions $\{f_i : Y_i \to X : i \in I\}$ is

$$Z = \{z \in \prod_I Y_i : f_i(z_i) = f_j(z_j) \text{ for all } i, j \in I\},$$

with projection maps $p_i(z) = z_i$ for $i \in I$. Note that Z is a closed subspace of $\prod_I Y_i$ so that Z is also the pullback in $\mathbf{K}$, the category of compact spaces.

PROPOSITION 3.1: The category of flows has pullbacks.

Proof: Given a set $\{f_i : Y_i \to X : i \in I\}$ of flow maps, let

$$\{p_i : Z \to Y_i : i \in I\}$$

be the pullback in spaces. In order to make the projections into flow maps we must define the action of $t \in T$ on Z to be

$$(ty)_i = ty_i.$$

Then it is routine to verify that Z is a flow, and that the p_i's are flow maps which inherit their universality in flows from their universality in spaces. ❑

The relevance of perfect maps to our enterprise is provided by the following observation.

PROPOSITION 3.2: For every perfect flow surjection $f : Y \to X$ there is a closed subflow $Y_0 \subseteq Y$ such that the restriction of f to Y_0 is perfect and T-irreducible.

Proof: Let S be the collection of all closed subflows $S \subseteq Y$ such that $f(Y) = X$. We claim that S is closed under the intersection of chains. For if $S_0 \subseteq S$ is totally ordered by inclusion, and if $\cap S_0 = S_0$, then S_0 is at least a closed subflow of Y. Furthermore, $f(S_0) = X$ because for each $x \in X$ the compact set $f^{-1}(x)$ meets each of the closed sets $S \in S_0$ nontrivially, and therefore meets S_0 nontrivially. Finally, it is clear that the restriction of a perfect map to a closed subspace is perfect. ❑

The pullback of perfect continuous functions has projections which are perfect continuous functions. This result is folklore; we prove it here in the interests of a self-contained treatment. It requires a preliminary lemma.

LEMMA 3.3: Suppose we have a collection $\{Y_i : i \in I\}$ of spaces, and for each $i \in I$ a compact subspace $C_i \subseteq Y_i$. Let Y denote $\prod_I Y_i$ with projection maps $p_i : Y \to Y_i$, and let $C = \prod_I C_i$. Then any neighborhood of C contains a neighborhood of the form $\bigcap_{I_0} p_i^{-1}(U_i)$ for some finite $I_0 \subseteq I$ and collection $\{U_i : i \in I_0\}$, each U_i a neighborhood of C_i in Y_i.

Proof: We prove this lemma first for finite index sets by induction on $|I|$. If $|I| = 1$ we can take $U_1 = U$. Assume now that the lemma holds for index sets of cardinality n, and consider a collection of spaces $\{Y_i : 1 \le i \le n+1\}$ with compact subspaces $\{C_i : 1 \le i \le n+1\}$, and neighborhood U of $\prod_{1 \le i \le n+1} C_i$. Let $Y = \prod_{1 \le i \le n} Y_i$ and $C = \prod_{1 \le i \le n} C_i$. We claim that for each $c \in C$ there are neighborhoods $W(c)$ of c in Y and $V(c)$ of C_{n+1} in Y_{n+1} such that $W(c) \times V(c) \subseteq U$. This is true because for each $c_{n+1} \in C_{n+1}$ there are neighborhoods $R(c_{n+1})$ of c and $S(c_{n+1})$ of c_{n+1} such that $R(c_{n+1}) \times S(c_{n+1}) \subseteq U$; since C_{n+1} is compact, a finite number of the $S(c_{n+1})$'s cover it, and we may take $V(c)$ to be the union of these $S(c_{n+1})$'s and $W(c)$ to be the intersection of the corresponding $R(c_{n+1})$'s. We next claim that there are neighborhoods W of C in Y and V of C_{n+1} in Y_{n+1} such that $W \times V \subseteq U$. This is true because C is compact and therefore covered by a finite number of W(c)'s; take W to be the union of these $W(c)$'s and V to be the intersection of the corresponding $V(c)$'s. Finally, use the induction hypothesis to get neighborhoods U_i of C_i in Y_i for $1 \le i \le n$ such that $\prod_{1 \le i \le n} U_i \subseteq U$, and set $U_{n+1} = V$. Then

$$\prod_{1 \le i \le n+1} C_i \subseteq \prod_{1 \le i \le n+1} U_i \subseteq U.$$

If the index set is infinite, then because C is compact, any neighborhood of it contains a neighborhood which is a finite union of basic open sets in the product. But such a neighborhood depends on only a finite set of indices, and so the argument of the previous paragraph applies and yields the desired conclusion. ❑

PROPOSITION 3.4: If $\{f_i : i \in I\}$ is a set of perfect surjections with common codomain, then its pullback $\{p_i : i \in I\}$ in spaces is a set of perfect surjections, and $f = f_i p_i$ (independent of $i \in I$) is perfect.

Proof: Let Y_i denote the domain of f_i and X its codomain. Consider the pullback Z to be a subspace of $Y = \prod_I Y_i$ as above. Observe first that for any point $x \in X$, $f^{-1}(x)$ is homeomorphic to the compact space $\prod_I f_i^{-1}(x)$. To show f closed consider a closed subspace $D \subseteq Z$ and point $x \in X \backslash f(D)$. By Lemma 3.3 applied to the neighborhood $Y \backslash D$ of $f^{-1}(x)$, there is a finite set $I_0 \subseteq I$ and collection $\{U_i : i \in I_0\}$, each U_i an open neighborhood of $f_i^{-1}(x)$, such that

$$f^{-1}(x) \subseteq \bigcap_{I_0} p_i^{-1}(U_i) \subseteq X \backslash D.$$

But for each $i \in I_0$ the closure of f_i implies that

$$V_i = \{x' \in X : f_i^{-1}(x') \subseteq U_i\} = X \backslash f_i(Y_i \backslash U_i)$$

is a neighborhood of x. But then $V = \bigcap_{I_0} V_i$ constitutes a neighborhood of x which is disjoint from $f(C)$, for if $f(y) = x' \in V$ then $y_i \in f_i^{-1}(x') \subseteq U_i$ for all $i \in I_0$, meaning $y \notin C$. This completes the proof that f is perfect, from which it follows that each f_i and p_i is perfect. ❑

The pullback of T-irreducible maps typically has projections which are not T-irreducible. Nevertheless, by Proposition 3.2 we can find a subflow of the pullback on which the restrictions of the projections are T-irreducible. What we lose in passing to the subflow is the universal property of pullbacks.

PROPOSITION 3.5: Given a set $\{f_i : i \in I\}$ of perfect T-irreducible maps with common codomain there is a set $\{p_i' : i \in I\}$ of perfect T-irreducible maps with common domain such that $f_i p_i' = f_j p_j'$ for all $i, j \in I$. In this case $f = f_i p_i$ is also perfect and T-irreducible.

Proof: Let Y_i denote the domain of f_i, $\{p_i : i \in I\}$ the pullback in flows, and Z the common domain of the p_i's. Then use Proposition 3.2 to find a subflow Z_0 and insertion map $j : Z_0 \to Z$ such that fj is T-irreducible. Then the desired maps are $p_i' = p_i j$. ❑

The next ingredient in the proof is a cardinality bound on the T-irreducible preimages of a given flow. The ideas involved in this observation are a special case of the more general Wretched Diagram Theorem to come in Section 6.3.

PROPOSITION 3.6: For any T-irreducible map $f : Y \to X$,

$$|Y| \le 2^{2^{\kappa} \cdot |T|},$$

where κ is the least cardinality of a subspace $X_0 \subseteq X$ for which

$$TX_0 \equiv e(T \times X_0) = \{tx : t \in T, \quad x \in X_0\}$$

is dense in X.

Proof: Let X_0 be a subspace of X of cardinality κ such that TX_0 is dense. For each $x \in X_0$ choose $y_x \in Y$ such that $f(y_x) = x$, let $Y_X = \{y_x : x \in X_0\}$, and let $Y' = TY_X$. Then Y' is a subflow of Y which maps onto TX_0, and its cardinality is bounded by $\kappa \cdot |T|$. Since $\mathrm{cl}\, Y'$ is a closed subflow which maps onto X, it follows from the T-irreducibility of f that Y' is dense in Y. Since each element in Y can then be associated with the trace of its neighborhood filter on Y', the result follows. ❑

3.2. The Projective Cover

We have now assembled the machinery necessary to prove the existence of the projective cover of a flow. We shall need the following simple tree lemma. A *for-*

est is a partially ordered set in which the set of predecessors of any element is well-ordered. A *tree* is a forest with a least element called the *root*, and any forest is a union of trees. The *level* of an element of a forest F is the order type of its predecessors, and the *a-th level of F* is the set of elements of level α. The *height of F* is the least ordinal greater than the levels of all its elements. We say that F *branches at level* α if there is some $a \in F$ of level α for which there are at least two successors of level $\alpha+1$. A *path* through F is a maximal totally ordered subset. We leave the straightforward inductive proof of the following lemma to the reader.

LEMMA 3.7: If a forest of height κ has branching at every level, then there are at least $|\kappa|$ paths through it.

THEOREM 3.8: Every flow has a perfect T-irreducible preimage which itself has no proper perfect T-irreducible preimage.

Proof: If a flow X had no such preimage, then we claim that we could find perfect T-irreducible preimages of X of arbitrary cardinality as follows. The idea is to inductively define proper perfect T-irreducible maps $p^\alpha : Y_\alpha \twoheadrightarrow X$ for each ordinal α, and to simultaneously define perfect T-irreducible surjections $p_\beta{}^\alpha : Y_\alpha \twoheadrightarrow Y_\beta$ for all $\beta < \alpha$, preserving as we do so the compatibility conditions

$$p_\gamma{}^\beta p_\beta{}^\alpha = p_\gamma{}^\alpha \quad \text{and} \quad p^\beta p_\beta{}^\alpha = p^\alpha \quad \text{for all} \quad \alpha > \beta > \gamma.$$

Let p^0 be the identity map on $Y_0 = X$, and suppose that $p_\gamma{}^\beta$ and p^β have been defined for all $\alpha > \beta > \gamma$. If $\alpha = \beta + 1$ then, since Y_β is a perfect T-irreducible preimage of X, it has a proper perfect T-irreducible preimage, and we choose any one to be $p_\beta{}^\alpha : Y_\alpha \twoheadrightarrow Y_\beta$. Then define $p^\alpha = p^\beta p_\beta{}^\alpha$, and define $p_\gamma{}^\alpha = p_\gamma{}^\beta p_\beta{}^\alpha$ for all $\beta > \gamma$. If α is a limit ordinal then let $\{p_\beta{}^\alpha : \beta < \alpha\}$ be the perfect T-irreducible maps with common domain Y_α given by Proposition 3.5 from the set $\{p^\beta : \beta < \alpha\}$, and set $p^\alpha = p^\beta p_\beta{}^\alpha$, independent of $\beta < \alpha$.

To complete the argument note that the cardinality of Y_α is at least that of α. For $F = \bigcup_{\beta < \alpha} Y_\beta$ carries a natural forest ordering given by $y_\gamma < y_\beta$ if and only if $y_\gamma \in Y_\gamma$, $y_\beta \in Y_\beta$, $\gamma < \beta$, and $p_\gamma{}^\beta(y_\beta) = y_\gamma$. Each path through this forest corresponds to a nested collection of compact subsets of Y_α, and this collection has nonempty intersection in Y_α. Furthermore, distinct paths contain elements corresponding to disjoint sets in Y_α, and so have intersections which are disjoint in Y_α. The cardinality claim then follows from Lemma 3.7. But the claim itself is inconsistent with the cardinality bound of Proposition 3.6. ❑

The next order of business is to prove the uniqueness of the projective cover of a flow. We shall call two flow surjections $f_i : Y_i \twoheadrightarrow X$ *equivalent* if there is a flow homeomorphism $g : Y_1 \twoheadrightarrow Y_2$ such that $f_1 = f_2 g$.

THEOREM 3.9: For every flow X there is a flow map $f : Y \twoheadrightarrow X$ which is unique up to equivalence with respect to the following equivalent properties.

(1) A flow map $g : Z \twoheadrightarrow X$ is perfect and T-irreducible if and only if $f = gk$ for some flow surjection k.

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ X & \end{tikzcd}$$

(2) f is perfect and T-irreducible, and Y has no proper perfect T-irreducible preimages.

(3) f is perfect and T-irreducible, and Y is a retract of any space which maps perfectly onto it. That is, for any perfect flow surjection Y there is a perfect flow embedding $i : Y \to Z$ such that $gi = 1_Y$.

(4) f is perfect and T-irreducible, and Y is a projective in the category of flows and perfect flow morphisms. That is, for any perfect flow maps q and g such that g is surjective there is a (perfect) flow morphism k such that $gk = q$.

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ Y \arrow[ur, "k"] \arrow[r, "q"'] & W \end{tikzcd}$$

Proof: We established the existence of a preimage satisfying (2) in Proposition 3.8, so it remains only to prove the equivalence of the conditions. Suppose that f satisfies (1). We know from Proposition 3.8 that X has a perfect T-irreducible preimage $g : Z \to X$ which itself has no proper perfect T-irreducible preimage, and so $f = gk$ for some flow surjection k, which must be perfect and T-irreducible. It follows that Y is equivalent to Z, and we conclude that f and Y satisfy (2). If f and Y satisfy (2), and if $g : Z \to Y$ is a perfect surjection, then by Proposition 3.2 there is a closed invariant subspace $Z_0 \subseteq Z$ such that $g|Z_0$ is a perfect T-irreducible surjection. But then $g|Z_0$ is a homeomorphism, and the desired embedding is its inverse.

Now assume (3), and suppose that we are given perfect flow maps g and q as in (4). Let V be the pullback of g and q with projection maps p_Z and p_Y, so that $gp_Z = qp_Y$. Then these projections are perfect by Proposition 3.4, and p_Y is surjec-

$$\begin{tikzcd} V \arrow[r, "p_Z"] \arrow[d, shift right, "p_Y"'] & Z \arrow[d, "g"] \\ Y \arrow[u, shift right, "i"'] \arrow[ur, "k"] \arrow[r, "q"'] & W \end{tikzcd}$$

tive because for any $y \in Y$ there is some $z \in Z$ such that $g(z) = q(y)$, hence $v =$

$(y, z) \in V$ and $p_Y(v) = y$. Therefore, by (3) there is an embedding i such that $p_Y i = 1_Y$. Then we have the map $k = p_Z i$ such that

$$gk = gp_Z i = qp_Y i = q,$$

which shows that Y is projective. To complete the proof observe that the nontrivial implication in (1) is a special case of (4). ❑

4. UNIVERSAL MINIMAL FLOWS

In this very brief section we show that the existence and uniqueness of universal minimal flows follows easily from the considerations of Section 3 above. In this section T is an arbitrary topological monoid. Following Auslander [1], we call a flow $X \in \mathbf{TK}$ *minimal* if for all $x \in X$, $Tx = \{tx : t \in T\}$ is dense in X. A minimal flow $Y \in \mathbf{TK}$ is a *universal minimal flow* if, for every minimal flow X, there is a surjective flow map $f : Y \to X$.

Let $\{\bullet\}$ be a one point space, and let T act on $\{\bullet\}$ in the only way possible, namely trivially. The following two easy results establish the connection between projectives and universal minimal flows, and provide an alternate method of proving the existence and uniqueness of the universal minimal flows. The proofs are straightforward so we omit them.

LEMMA 4.1: Let T be a topological monoid and $X \in \mathbf{TK}$. Then the following are equivalent.

(1) X is a minimal flow.
(2) The collapsing map $c : X \to \{\bullet\}$ is T-irreducible.

THEOREM 4.2: Let T be a topological monoid and $X \in \mathbf{TK}$. Then the following are equivalent.

(1) X is the unique universal minimal flow.
(2) X is the T-projective cover of $\{\bullet\}$.

We denote the space X satisfying Theorem 4.2 as $\hat{T}$.

5. EXAMPLES OF PROJECTIVE COVERS

At this point we have established the existence and uniqueness of the projective cover in the category $\mathbf{TTych^P}$. We have also provided a characterization of projectives in $\mathbf{TK}$ in Theorem 2.6 as being precisely the flows of the form $T \times \Omega$, where Ω is an extremally disconnected compact space, and where T acts on the left factor by left multiplication and the right factor trivially. Here we provide examples which show that certain natural conjectures about projectives fail when the assumptions on T are relaxed somewhat. The techniques used in this section are ad hoc. We begin to remedy this example-by-example approach in Section 6.3 below.

EXAMPLE 5.1: The group assumption is necessary in Theorem 2.6.

Proof: Let $T = \{1, t\}$, with unit 1 and $t^2 = t$, with discrete topology. Let T act trivially on the one point space $\{\bullet\}$. Let $f : Y \to \{\bullet\}$ be T-irreducible and $y \in Y$. Then it is easy to see that $\{ty\}$ is a closed subflow of Y for which $f\{ty\} = \{\bullet\}$. Since f is T-irreducible, $Y = \{ty\}$, so Y is a one point space. Observe that, by way of comparison, if the *group* $\mathbb{Z}_2$ acts trivially on $\{\bullet\}$, the projective cover is $\mathbb{Z}_2 \times \{\bullet\}$ or just $\mathbb{Z}_2$. ❑

The next example is considerably more important and illustrative, because the monoid T is the discrete group of integers $\mathbb{Z}$, the space X is compact, yet the projective cover is not a product. Of course, action by $\mathbb{Z}$ represents forward and reverse iteration by a nonperiodic homeomorphism. Before presenting the example, we need one (well-known) lemma and a (well-known) consequence.

LEMMA 5.2: Let X be compact, and let $j : X \times \mathbb{Z} \to \beta(X \times \mathbb{Z})$ be the natural injection. Suppose that X contains a countable set $\{x_n : n \in \mathbb{Z}\}$ and an element $\bar{x}$ which is an accumulation point of the x_n's such that $x_n \neq \bar{x}$ for all n. Then

$$\{j(\bar{x}, n) : n \in \mathbb{Z}\}' \cap \{j(x_n, n) : n \in \mathbb{Z}\}' = \varnothing.$$

Here E' denotes the set of accumulation points of the set E.

Proof: For each n, let $f_n : X \to \mathbb{R}$ be a continuous function satisfying $0 \le f_n \le 1$, $f_n(\bar{x}) = 1$ and $f_n(x_n) = 0$. Define $F : X \times \mathbb{Z} \to \mathbb{R}$ by $F(x, n) = f_n(x)$. Observe that $F(\bar{x}, n) = 1$ and $F(x_n, n) = 0$ for all n. In particular, after extending F to $\beta(X \times \mathbb{Z})$, we must have $F(v) = 1$ if $v \in \{j(\bar{x}, n) : n \in \mathbb{Z}\}'$, while $F(w) = 0$ if $w \in \{j(x_n, n) : n \in \mathbb{Z}\}'$. ❑

COROLLARY 5.3: Let X be an infinite compact space. Then $\beta(X \times \mathbb{Z})$ and $X \times \beta\mathbb{Z}$ are not homeomorphic. Also, there is a continuous surjection $\beta : \beta(X \times \mathbb{Z}) \to X \times \beta\mathbb{Z}$.

Proof: (See also [34, Chapter 8]). Consider the following diagram. Here, i

$$\begin{tikzcd}
\beta(X \times \mathbb{Z}) \arrow[dr, "\beta"] & \\
X \times \mathbb{Z} \arrow[u, "j"] \arrow[r, "i"'] & X \times \beta\mathbb{Z}
\end{tikzcd}$$

and j are the natural injections of $X \times \mathbb{Z}$ into $X \times \beta\mathbb{Z}$ and $\beta(X \times \mathbb{Z})$, respectively, and β is the extension of i. Select a countable set in X and an accumulation point of this set which satisfy the assumptions of Lemma 5.2. Then observe that the function F from Lemma 5.2 does not extend to a continuous function on $X \times \beta\mathbb{Z}$. ❑

EXAMPLE 5.4: If T acts trivially on X in **TK**, and X is extremally disconnected, then $\gamma^T X$ need not be $T \times X$.

Recall that $\hat{\mathbb{Z}}$ denotes the $\mathbb{Z}$-projective cover of the one point space $\{\bullet\}$, where $\mathbb{Z}$ has the discrete topology (Theorem 4.2). Now let $\mathbb{Z}$ act trivially on the space $\beta\mathbb{Z}$, on $\hat{\mathbb{Z}} \times \beta\mathbb{Z}$ by the rule $z(x, y) = (zx, y)$ for $z \in \mathbb{Z}$ and $(x, y) \in \hat{\mathbb{Z}} \times \beta\mathbb{Z}$, and on $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$ by the continuation of the actions on $\hat{\mathbb{Z}} \times \beta\mathbb{Z}$. Since $\mathbb{Z}$ is discrete, evaluation of these actions is continuous and these spaces are in fact flows. Then one easily establishes the following claims, whose proofs are briefly sketched below.

(1) The projection map $p : \hat{\mathbb{Z}} \times \beta\mathbb{Z} \to \beta\mathbb{Z}$ is $\mathbb{Z}$-irreducible.

(2) The injection $i : \hat{\mathbb{Z}} \times \mathbb{Z} \to \hat{\mathbb{Z}} \times \beta\mathbb{Z}$ extends to a flow map $\beta : \beta(\hat{\mathbb{Z}} \times \mathbb{Z}) \to \hat{\mathbb{Z}} \times \beta\mathbb{Z}$. The map β is $\mathbb{Z}$-irreducible and is **not** a homeomorphism.

(3) $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$ is the $\mathbb{Z}$-projective cover of $\beta\mathbb{Z}$.

Proof: (1) It is clear that p is a flow map. If K is a closed subflow of $\hat{\mathbb{Z}} \times \beta\mathbb{Z}$ such that $p(K) = \beta\mathbb{Z}$ then for each $y \in \beta\mathbb{Z}$ there is an $x \in \hat{\mathbb{Z}}$ such that $(x, y) \in K$. Then $\mathbb{Z}(x, y) = (\mathbb{Z}x, y)$ is a subset of K which is dense in $\hat{\mathbb{Z}} \times \{y\}$.

(2) The first part of the assertion is routine: since $\mathbb{Z}$ is discrete one need only verify that $\mathbb{Z}$ acts on $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$. Also, since $i(\hat{\mathbb{Z}} \times \mathbb{Z})$ is open and dense in $\hat{\mathbb{Z}} \times \beta\mathbb{Z}$, β is $\mathbb{Z}$ -irreducible. And since the spaces $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$ and $\hat{\mathbb{Z}} \times \beta\mathbb{Z}$ are not homeomorphic, Corollary 5.3 (with $X = \hat{\mathbb{Z}}$) implies that β is not a homeomorphism. (See also [34, Chapter 8]).

(3) Let Y be a compact flow and $f : Y \to \beta(\hat{\mathbb{Z}} \times \mathbb{Z})$ a $\mathbb{Z}$-irreducible map. For each $z \in \mathbb{Z}$ let Y_z denote $f^{-1}(\hat{\mathbb{Z}} \times \{z\})$ and f_z the restriction of f to Y_z. Observe that each Y_z is a clopen subflow of Y on which f_z is $\mathbb{Z}$-irreducible, hence a flow homeomorphism. Let $g_z : \hat{\mathbb{Z}} \times \{z\} \to Y_z$ be the inverse of f_z, and define $g : \beta(\hat{\mathbb{Z}} \times \mathbb{Z}) \to Y$ by first defining g on $\hat{\mathbb{Z}} \times \mathbb{Z}$ by $g(w, z) = g_z(w)$ and then extending it to $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$. Then g commutes with the actions because each g_z does, and since fg is the identity on the dense subset $\hat{\mathbb{Z}} \times \mathbb{Z} \subseteq \beta(\hat{\mathbb{Z}} \times \mathbb{Z})$, it follows that fg is the identity map. Thus $\beta(\hat{\mathbb{Z}} \times \mathbb{Z})$ is the $\mathbb{Z}$-projective cover of $\beta\mathbb{Z}$ by Theorem 3.9(3). ❑

The existence of a minimal dense open subflow of $\beta\mathbb{Z}$ is the crucial feature of the above example. By arguing exactly as in Example 5.4, one can establish the following theorem. We are obliged to confess, however, that we are at present unable to characterize the $\mathbb{Z}$ -projective cover of an arbitrary compact flow X on which $\mathbb{Z}$ acts trivially.

THEOREM 5.5: Suppose that T is a discrete monoid which acts trivially on the compact flow X, and suppose further that X has a dense discrete subspace S. Then

$$\gamma^T X = \beta(\hat{T} \times S).$$

It is instructive to attempt to replicate the process used in Example 5.4 when $\mathbb{Z}$ is replaced by an infinite compact group T acting trivially on $\beta\mathbb{Z}$. After all, we have shown that in this case the projective cover is $T \times \beta\mathbb{Z}$ and not $\beta(T \times \mathbb{Z})$. So suppose that $\beta\mathbb{Z}$ is a flow on which T acts trivially. It is easy to see (Corollary 5.3) that $\beta(T \times \mathbb{Z}) \neq T \times \beta\mathbb{Z}$, T acts on $\beta(T \times \mathbb{Z})$, and there is a surjective map $\beta(T \times \mathbb{Z}) \to T \times \beta\mathbb{Z}$ which respects the actions by T.

However, $\beta(T \times \mathbb{Z})$ is *not* a flow. Indeed, let (t_α, n_α) be a net in $T \times \mathbb{Z}$ such that $t_\alpha \to 1$, $t_\alpha \neq 1$ for all α and $n_\alpha \to z \in \beta(\mathbb{Z})\backslash\mathbb{Z}$. To see that such a net exists, observe that there is a countable set $\{t_n' : n \in \mathbb{Z}\}$, $t_n' \neq 1$ for all n, which clusters at $1 \in T$. Let $\{t_\alpha : \alpha \in D\}$ be a net in this set converging to 1. Put $w_n' = (t_n', n) \in \beta(T \times \mathbb{Z})$ and let $w_\alpha = (t'_{n_\alpha}, n_\alpha)$ be the corresponding net in $\beta(T \times \mathbb{Z})$. Pass to a subnet if necessary (and rename the directed set) so that $n_\alpha \to z \in \beta\mathbb{Z}$. It is immediate that $z \in$

$\beta(\mathbb{Z})\backslash\mathbb{Z}$. The argument given in Lemma 5.2 (with $X = T$) shows that $(t_\alpha, n_\alpha) \not\to w$ in $\beta(T \times \mathbb{Z})$, where $w = \lim(1, n_\alpha)$. So, $t_\alpha \to 1$ and $(1, n_\alpha) \to w$, but

$$t_\alpha(1, n_\alpha) = (t_\alpha, n_\alpha) \not\to 1w = w.$$

This shows that the evaluation map is not continuous and that $\beta(T \times \mathbb{Z})$ is not a flow.

EXAMPLE 5.6: Here is a brief example which further illustrates the difficulty of characterizing the projective cover. The details of this example are given in the paper [8].

Let Π be the group of permutations of the integers $\mathbb{Z}$ under composition. Topologize Π by using as neighborhoods of $\pi \in \Pi$ sets of the form

$$N_m(\pi) = \{\pi' : \pi'(i) = \pi(i) \ \forall i, -m \le i \le m\}$$

for $m \in \mathbb{N}$. With this topology, Π is a topological group. Let $\sigma: \mathbb{Z} \to \mathbb{Z}$ be given by $\sigma(i) = i + 1$. Then we show using Ramsey's theorem [31] that σ has fixed points when Π acts on $\hat{\Pi}$. (Recall that $\hat{\Pi}$ is the projective cover of the trivial action of Π on a one point space.) The point here is that even though $\hat{\Pi}$ is a projective flow, the action of Π is not free in the sense of Auslander [1], i.e., some of the permutations have fixed points. This again shows that a characterization of the projective cover cannot be simple.

6. ANOTHER CONSTRUCTION OF THE PROJECTIVE COVER

6.1. The Free Flow Over a Space

Every space can be embedded in a flow "as freely as possible," and uniquely at that. Given a space X, let τX denote the flow $T \times X$, with actions defined by the rule $t(t', x) = (tt', x)$. Let $\tau_X: X \to T \times X$ be the embedding given by $\tau_X(x) = (1, x)$. Then τX is the free flow over the space X in the following sense.

PROPOSITION 6.1: For any space X there is a flow τX and an embedding τ_X such that any continuous function f from X into a flow Y extends uniquely to a flow morphism τf which makes this diagram commute.

$$\begin{tikzcd}
\tau X \arrow[dr, "\tau f"] & \\
X \arrow[u, "\tau_X"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof: If τf is to commute with the actions then we must have

$$\tau f(t, x) = (\tau f)\, t(1, x) = (\tau f)\, t\tau_X(x) = t(\tau f)\, \tau_X(x) = tf(x).$$

But if we take this as the definition of τf then it is easy to check that it is a flow map. ❑

6.2. The Stone-Čech Compactification of a Flow

A crucial feature in the construction in Section 6.3 is the appropriate generalization of the idea of the Stone-Čech compactification to the context of flows. This topic is treated in detail in the paper [6]. Here we content ourselves with a brief discussion of the results from that paper which are needed for this one.

Let X be a flow in **TTych** and $f : X \to \mathbb{R}$ a bounded continuous function. We say that f is *T-uniformly continuous* if for every $\varepsilon > 0$ and $t \in T$ there exists a neighborhood T_t of t such that, for all $t' \in T_t$ and $x \in X$,

$$|f(t'x) - f(tx)| < \varepsilon.$$

One immediately establishes the following.

LEMMA 6.2: Every bounded continuous real-valued function on a compact flow is T-uniformly continuous.

Let $C^T X$ denote the set of bounded T-uniformly continuous real-valued functions on the flow X. To construct the Stone-Čech compactification of a compact flow X, consider the set

$$\mathcal{F} = \{f \in C^T X : 0 \le f(x) \le 1 \ \ \forall x \in X\}.$$

Consider the map

$$i : X \to [0, 1]^{\mathcal{F}}$$

defined by $(ix)_f = f(x)$ for $f \in \mathcal{F}$. Define the Stone-Čech compactification of the flow X by setting

$$\beta^T X = \mathrm{cl}(iX),$$

where cl denotes the closure in $[0,1]^{\mathcal{F}}$. Let us also define the map $\beta_X{}^T : X \to \beta^T X$ in the obvious way, namely as the codomain restriction of i. The next result shows that the Stone-Čech compactification possesses the expected universal properties.

LEMMA 6.3: Let X and Y be Tychonoff flows.

(1) Then $\beta^T X$ is a compact flow and $\beta_X{}^T : X \to \beta^T X$ is a flow map onto a dense subflow of $\beta^T X$.
(2) If X is compact then $\beta_X{}^T : X \to \beta^T X$ is a surjective flow homeomorphism. We write (by abuse of notation) $\beta^T X = X$.
(3) Let $f : X \to Y$ be a flow map. Then f lifts to a unique flow map $\beta^T f : \beta^T X \to \beta^T Y$ such that the following diagram commutes:

$$\begin{tikzcd} \beta^T X \arrow[r, "\beta^T f"] & \beta^T Y \\ X \arrow[u, "\beta^T_X"] \arrow[r, "f"'] & Y \arrow[u, "\beta^T_Y"'] \end{tikzcd}$$

We call a flow X *compactifiable* if the map $\beta_X{}^T$ is a homeomorphic embedding. Part (2) of this lemma says that compact flows are compactifiable. But in general the map $\beta_X{}^T$ need not be injective; there are flows (even in **TTych**) which are not flow homeomorphic to a dense subflow of a compact flow. In other words, there are noncompactifiable flows. (See [6] for examples.) For us, the following result suffices.

LEMMA 6.4: The following are equivalent for a flow X.
(1) X is compactifiable.
(2) $\beta_X{}^T$ is a bicontinuous injection, i.e., a subspace insertion.
(3) C^TX separates the points of X and determines the topology of X.

A final result in this section states that topological groups are well behaved with respect to compactifiability. The proof, given in [6, Corollary 6.4], relies on the fact that certain pseudometrics on T can be shown to be T-uniformly continuous.

PROPOSITION 6.5: A topological group, considered as a flow acting on itself by left multiplication, is compactifiable.

6.3. The Wretched Diagram

We can construct the projective cover of a compact flow X as follows. Let $i: S \to X$ be an injection from a discrete space S onto a subspace $i(S) \subseteq X$ with the property that $Ti(S) = \{ti(s) : s \in S,\ t \in T\}$ is dense in X. Let τS be the free flow over S, let $\beta^T\tau S$ be the Stone-Čech compactification of τS, and let Y be a closed subflow of $\beta^T\tau S$ minimal with respect to satisfying $\beta^T\tau i(Y) = X$ (Proposition 3.2). Let k denote the insertion of Y in $\beta^T\tau S$ and let f denote $(\beta^T\tau i)k$. This is summarized in the following diagram, which is that part of the wretched diagram used to construct the projective cover.

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "\tau_S"'] & X & Y \arrow[l, "f"'] \arrow[d, "k"] \\ \tau S \arrow[ur, "\tau i"] \arrow[rr, "\beta^T_{\tau S}"'] & & \beta^T\tau S \arrow[ul, "\beta^T\tau i"'] \end{tikzcd}$$

THEOREM 6.6: Y is the projective cover of X.

Proof: To verify that $f: Y \rightarrow X$ is the projective cover of X consider another perfect T-irreducible surjection $g: Z \rightarrow X$. The full-blown Wretched Diagram summarizes the entire construction.

$$\begin{tikzcd}
 & Z \arrow[d, "g"] & \\
S \arrow[ur, "j"] \arrow[r, "i"] \arrow[d, "\tau s"'] & X & Y \arrow[l, "f"'] \arrow[ul, "h"'] \arrow[d, "k"] \\
\tau S \arrow[ur, "\tau i"] \arrow[rr, "\beta^T_{\tau S}"'] \arrow[uur, bend left=70, "\tau j"] & & \beta^T\tau S \arrow[ul, "\beta^T\tau i"'] \arrow[uul, bend right=70, "\beta^T\tau j"']
\end{tikzcd}$$

Because discrete spaces are projectives in spaces, we can find a map j satisfying $gj = i$, and j induces τj by the freeness of τS. Since Z is necessarily compact, τj in turn induces $\beta^T\tau j$. The desired map $h = (\beta^T\tau j)k$ is surjective because g is T-irreducible and the entire diagram commutes. ❑

Several remarks are in order.

(1) The only relevant property of S, aside from the density of $Ti(S)$, is its projectivity in spaces. But in fact it is enough for S to be projective in **K**. We could, for example, take S to be the classical (no action) Gleason cover of X.

(2) The construction works whether or not τS is compactifiable, i.e., whether or not $\beta^T\tau S$ is injective. But we remark in passing that τS is compactifiable if and only if T, acting on itself by left multiplication, is compactifiable. This is the case, for example, when T is a topological group by Proposition 6.5.

(3) The function of the passage to the Stone-Čech compactification is to restore the quality of perfection to the maps so as to be able to use Proposition 3.2. The maps on the left side of the diagram lack this quality, while the maps on the right side possess it.

(4) When we take discrete S (and the free flow τS) on the left side of the diagram, we "forget" the topology on X in the process. This seems perplexing at first, until we observe that all continuous functions on X give, under composition with τi, bounded T-uniformly continuous functions on τS. Thus $\beta^T\tau S$ carries enough information to "restore" the topology on X.

COROLLARY 6.7: In the terminology of Theorem 6.6, $\gamma^T X$ is a flow retract of $\beta^T\tau S$.

Proof: Replace Z by $\gamma^T X$ in the wretched diagram to get the flow surjection $\beta^T\tau j$, and then use Theorem 3.9(3) to get the desired conclusion. ❑

COROLLARY 6.8: In the terminology of Theorem 6.6, $\gamma^T X$ is extremally disconnected whenever $\beta^T \tau S$ is. In particular, if T is discrete then $\gamma^T X$ is extremally disconnected for all compact flows X.

Proof: If T is discrete then $\tau S = T \times S$ is discrete, hence $\beta^T \tau S = \beta \tau S$ is extremally disconnected. ❑

A special case of Corollary 6.8 was proven by Balcar and Franek in [3]; it is due to Ellis in the group case.

THEOREM 6.9: The universal minimal flow $\hat{T}$ of a discrete semigroup T is extremally disconnected.

We close this section by providing several examples which show how some of our previous results and some new ones can be derived by applications of the wretched diagram. In the first, we give another proof of Theorem 2.6, which we restate: Suppose that X is a compact flow acted upon by a compact group T. Then $\gamma^T X = T \times \gamma(X/T)$.

Proof: In the wretched diagram, replace X by X/T, with trivial action by T, and replace S by $\gamma(X/T)$. Then $\tau S = T \times \gamma(X/T)$ is already compact, so $\beta^T \tau S = \tau S$. Since the map $\beta^T \tau i$ is easily seen to be T-irreducible, Proposition 3.2 gives nothing new. We are done once we observe that any T-irreducible preimage of X is also a T-irreducible preimage of X/T. ❑

PROPOSITION 6.10: Let the discrete group $\mathbb{Z}$ act on $\beta\mathbb{Z}$ by addition. That is, extend the action of addition on $\mathbb{Z}$ to $\beta\mathbb{Z}$. Then $\gamma^{\mathbb{Z}} \beta\mathbb{Z} = \beta\mathbb{Z}$.

Proof: In the wretched diagram let $S = \{0\} \subseteq \beta\mathbb{Z}$ and observe that $\mathbb{Z}S = \mathbb{Z}$ is dense in $\beta\mathbb{Z}$. So $\tau S = \mathbb{Z} \times \{0\} = \mathbb{Z}$, and, as above, $\beta^{\mathbb{Z}} \mathbb{Z} = \beta\mathbb{Z}$. But now the map $\beta^{\mathbb{Z}} \tau i$: $\beta\mathbb{Z} \to \beta\mathbb{Z}$ is the identity, since it is the identity on the dense subspace $\mathbb{Z} \subseteq \beta\mathbb{Z}$. ❑

7. THE RECOGNITION PROBLEM

The central open problem remaining from the present work is this: If X is a compact flow, give necessary and sufficient conditions on X in order that $X = \gamma^T Y$ for some $Y \in \mathbf{TK}$. We have solved this "recognition problem" only in certain special cases.

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The Glicksberg Theorem on Weakly Compact Sets for Nuclear Groups

W. BANASZCZYK[a,c,e] AND E. MARTÍN-PEINADOR[b,d,e]

[a]*Institute of Mathematics*
Łódź University
Banacha 22
90–238 Łódź, Poland
and
[b]*Facultad de Ciencias Matemáticas*
Universidad Complutense de Madrid
28040 Madrid, Spain

ABSTRACT: By the weak topology on an Abelian topological group we mean the topology induced by the family of all continuous characters. A well-known theorem of I. Glicksberg says that weakly compact subsets of locally compact Abelian (LCA) groups are compact. D. Remus and F.J. Trigos-Arrieta [1993. Proceedings Amer. Math. Soc. **117**] observed that Glicksberg's theorem remains valid for closed subgroups of any product of LCA groups. Here we show that, in fact, it remains valid for all nuclear groups, a class of Abelian topological groups introduced by the first author in the monograph, "Additive subgroups of topological vector spaces" [1991. Lecture Notes in Math. **1466**].

There are several theorems in commutative harmonic analysis which remain valid for certain Abelian topological groups which are not locally compact. For instance, the Bochner theorem on positive-definite functions is true for nuclear locally convex spaces (see [6, Chapter 4, Section 2.3]), while the Pontryagin duality theorem is true for closed subgroups of countable products of locally compact Abelian (LCA) groups (see, e.g., [2] for further references). To treat results of this type from a unified point of view, the first author introduced in [1] the so-called nuclear groups, a class of Abelian topological groups which contains LCA groups and nuclear locally convex spaces, and is closed with respect to the operations of taking subgroups, separated quotients and arbitrary products (a different definition, of a nuclear Lie group, had been given in [6, Chapter 4, Section 5.4]).

Nuclear groups satisfy, among other properties, the Bochner theorem [1, Theorem 12.1] and, under some additional assumptions, also the Pontryagin duality theorem [1, Corollary 17.3]. From the point of view of convergent series and se-

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[c]E-mail: wbanasz@krysia.uni.lodz.pl.
[d]E-mail: peinador@sungt1.mat.ucm.es.
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quences, properties of nuclear groups are similar to those of nuclear spaces (see [1, Section 10] and [3]). For instance, every weakly convergent sequence of points of a nuclear group is convergent in the original topology [3, Theorem 1].

Let G be an Abelian topological group. By a *character* of G we mean a homomorphism of G into the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. By the *weak topology* on G we mean the topology induced by the family $G^\wedge$ of all continuous characters of G. Following [7], we say that G *respects compactness* if every weakly compact subset of G is compact in the original topology. It was proved in [7] that closed subgroups of any product of LCA groups respect compactness. The aim of this paper is to prove the following generalization of that result:

THEOREM: Nuclear groups respect compactness.

The proof given below is a modification of the proof of the above-mentioned Theorem 1 of [3]. We apply notation and terminology introduced in [3]. The family of neighborhoods of zero in an Abelian topological group G is denoted by $\mathcal{N}_0(G)$. Given a real number x, we denote by $\langle x\rangle$ the number $y \in (-½, ½]$ such that $x - y \in \mathbb{Z}$. For the definitions of a nuclear group and a nuclear vector group we refer the reader to [3] or to [1, (7.1) and (9.2)]. All vector spaces are assumed to be real.

LEMMA 1: Let $(x_s)_{s=1}^\infty$ be a sequence of nonzero real numbers with $|x_{s+1}/x_s| \geq 3$ for every s. Then there exists a real number t such that $|\langle tx_s\rangle| \geq \frac{1}{8}$ for every s.

Proof: For each $s = 1, 2, \ldots$, let $A_s = \{t \in \mathbb{R}: |\langle tx_s\rangle| \geq \frac{1}{8}\}$. We have to show that $\bigcap_{s=1}^\infty A_s \neq \emptyset$. All components of A_s are closed intervals of length $\frac{3}{4}x_s^{-1}$, hence all components of $\mathbb{R}\backslash A_s$ are open intervals of length $\frac{1}{4}x_s^{-1}$.

Now, choose any component I_1 of A_1. Since $|x_2/x_1| \geq 3$, it follows easily that I_1 must contain some component I_2 of A_2. Similarly, I_2 must contain some component I_3 of A_3, and so on. This allows us to construct inductively a decreasing sequence of closed intervals $(I_s)_{s=1}^\infty$ such that I_s is a component of A_s for every s; then $\bigcap_{s=1}^\infty A_s \supset \bigcap_{s=1}^\infty I_s \neq \emptyset$. ❑

Let $T: E \to F$ be a bounded linear operator acting between Banach spaces. By $d_k(T)$, $k = 1, 2, \ldots$, we denote the Kolmogorov numbers of T (see [9, p. 308]). The distance of a point $u \in F$ to a subset A of F is denoted by $d(u, A)$. By span$A$ we denote the linear subspace of F spanned over A. If K is an additive subgroup of E, then we denote by K_E^* the family of all continuous linear functionals f on E such that $f(K) \subset \mathbb{Z}$.

LEMMA 2: Let E, F be Hilbert spaces and $T: E \to F$ a bounded linear operator such that $\sum_{k=1}^\infty kd_k(T) \leq 1$. Let K be an additive subgroup of E. Given arbitrary $a \in E$ and $r > 0$ such that $d(Ta, T(K)) \geq r$, one can find an $f \in K_E^*$ with $|\langle f(a)\rangle| \geq \frac{1}{4}$ and $\|f\| \leq 4r^{-1}$.

This follows directly from Proposition (8.4) of [1]. The condition $\sum_{k=1}^\infty kd_k(T) \leq 1$ may be replaced by $\sum_{k=1}^\infty d_k(T) \leq c$, where c is some numerical constant; it is enough to apply Theorem 3.1(i) of [4] instead of Proposition (3.11) of [1] in the proof of (8.4) in [1].

LEMMA 3: Let $T: E \to F$ and $S: F \to G$ be bounded linear operators acting between Hilbert spaces. Suppose that $\sum_{k=1}^{\infty} k d_k(T) \leq 1$ and $d_k(S) \to 0$ as $k \to \infty$. Let K be an additive subgroup of E and $(a_n)_{n=1}^{\infty}$ a sequence in E such that

$$d(ST(a_m - a_n), ST(K)) \geq 1, \quad m \neq n. \tag{1}$$

Then one can choose a subsequence $(a_{n_s})_{s=1}^{\infty}$ of (a_n) satisfying the following condition: to each $u \in E$ there corresponds some $f \in K_E^*$ such that $|\langle f(u - a_{n_s})\rangle| \geq \frac{1}{8}$ for almost all s.

Proof: Suppose that

$$C := \sup_n d(Ta_n, T(K)) < \infty.$$

We can find a sequence $(v_n)_{n=1}^{\infty}$ in K such that $\| Ta_n - Tv_n \| < C + 1$ for every n. Then the set $\{ST(a_n - v_n)\}_{n=1}^{\infty}$ is totally bounded, because the condition $d_k(S) \to 0$ implies that S is a compact operator (see [9, p. 308]). Hence,

$$\liminf_{m,n \to \infty} \| ST(a_m - v_m) - ST(a_n - v_n) \| = 0,$$

which is impossible in view of (1). Thus $C = \infty$, and therefore, we may simply assume that $d(Ta_n, T(K)) \to \infty$ as $n \to \infty$.

The rest of the proof is similar to the proof of Lemma 3(b) in [3]. Let M be the linear subspace of E spanned over K, let N be the orthogonal completion of $\overline{M}$ in E, and let ϕ and ψ be the orthogonal projections of E onto $\overline{M}$ and N, respectively.

Suppose first that $\limsup \| \psi(a_n) \| = \infty$. Then there is a continuous linear functional g on N such that $\limsup |g\psi(a_n)| = \infty$ (weakly bounded subsets of locally convex spaces are bounded). We can choose a subsequence $(a_{n_s})_{s=1}^{\infty}$ of (a_n) such that $|g\psi(a_{n_{s+1}})/g\psi(a_{n_s})| \geq 4$ for every s. Now, take an arbitrary $u \in E$. Then

$$|g\psi(a_{n_{s+1}} - u)|/|g\psi(a_{n_s} - u)| \geq 3$$

for almost all s, say, for $s \geq s_0$. By Lemma 1, we can find some $t \in \mathbb{R}$ such that $|\langle tg\psi(a_{n_s} - u)\rangle| \geq \frac{1}{8}$ for $s \geq s_0$, and we may take $f = tg\psi$.

Next, suppose that $\limsup \| \psi(a_n) \| < \infty$. Choose a sequence $(b_n)_{n=1}^{\infty}$ in M with $b_n - \phi(a_n) \to 0$. For every n, we have

$$d(Ta_n, T(K)) \leq \| Ta_n - T\phi(a_n) \| + \| T\phi(a_n) - Tb_n \| + d(Tb_n, T(K)),$$

$$\| Ta_n - T\phi(a_n) \| \leq \| T \| \cdot \| a_n - \phi(a_n) \| = \| T \| \cdot \| \psi(a_n) \|,$$

$$\| T\phi(a_n) - Tb_n \| \leq \| T \| \cdot \| \phi(a_n) - b_n \|.$$

As $d(Ta_n, T(K)) \to \infty$, it follows that $d(Tb_n, T(K)) \to \infty$.

Choose an index n_1 such that $d(Tb_{n_1}, T(K)) > 2$. By Lemma 2, there is some $g_1 \in K_{\overline{M}}^*$ with $|\langle g_1(b_{n_1})\rangle| \geq \frac{1}{4}$ and $\| g_1 \| \leq 4 \cdot 2^{-1}$. As $b_{n_1} \in \operatorname{span} K$, we can find a finitely generated subgroup K_1 of K with $b_{n_1} \in M_1 := \operatorname{span} K_1$. Then we can find an index n_2 such that $d(Tb_{n_2}, T(K + M_1)) > 2^2$ and, by Lemma 2, some $g_2 \in (K + M_1)_{\overline{M}}^*$ with $|\langle g_2(b_{n_2})\rangle| \geq \frac{1}{4}$ and $\| g_2 \| \leq 4 \cdot 2^{-2}$. By repeating this procedure, we construct by induction a sequence $M_1 \subset M_2 \subset \ldots$ of finite-dimensional subspaces of M, a subsequence $(b_{n_s})_{s=1}^{\infty}$ of (b_n) and a sequence $g_s \in K_{\overline{M}}^*$ such that $b_{n_s} \in M_s$, $|\langle g_s(b_{n_s})\rangle| \geq \frac{1}{4}$, $g_{s+1}(M_s) = \{0\}$ and $\| g_s \| \leq 4 \cdot 2^{-s}$ for every s.

Now, take an arbitrary $u \in E$. We can find a positive integer p such that $\| \phi(u) \| \leq 2^{p-7}$ and $\| b_{n_s} - \phi(a_{n_s}) \| \leq 2^{p-7}$ whenever $s \geq p$. If $x, y \in \mathbb{R}$ and $|\langle y\rangle| \geq \frac{1}{4}$, then there is a coefficient $t = 0, \pm 1$ with $|\langle x + ty\rangle| \geq \frac{1}{4}$. Therefore, we can construct inductively a sequence $t_p, t_{p+1}, \ldots = 0, \pm 1$ such that

$$|\langle t_p g_p(b_{n_s}) + \ldots + t_s g_s(b_{n_s})\rangle| \geq \tfrac{1}{4}$$

for $s = p, p+1, \ldots$. Consider the functional $f_p = \sum_{r=p}^{\infty} t_r g_r$ on $\overline{M}$. It is clear that $f_p(K) \subset \mathbb{Z}$. We have $\|f_p\| \leq \sum_{r=p}^{\infty} \|g_r\| \leq 2^{-p+3}$. If $s \geq p$, then

$$|\langle f_p(b_{n_s})\rangle| = |\langle \textstyle\sum_{r=p}^{\infty} t_r g_r(b_{n_s})\rangle| = |\langle \textstyle\sum_{r=p}^{s} t_r g_r(b_{n_s})\rangle| \geq \tfrac{1}{4},$$

which implies that

$$\begin{aligned} |\langle f_p\phi(u - a_{n_s})\rangle| &= |\langle f_p(b_{n_s}) - f_p(b_{n_s}) + f_p\phi(a_{n_s}) - f_p\phi(u)\rangle| \\ &\geq |\langle f_p(b_{n_s})\rangle| - \|f_p\| \cdot \|b_{n_s} - \phi(a_{n_s})\| - \|f_p\| \cdot \|\phi(u)\| \geq \tfrac{1}{4} - \tfrac{1}{16} - \tfrac{1}{16} = \tfrac{1}{8}. \end{aligned}$$

So, we may take $f = f_p\phi$. ❑

Let p be a seminorm on a vector space E. We write $B_p = \{u \in E : p(u) \leq 1\}$. The quotient space $E/p^{-1}(0)$ endowed with its canonical norm is denoted by E_p, and the canonical projection of E onto E_p by ψ_p. We shall identify E_p with the corresponding subspace of the completion $\widetilde{E}_p$. We say that p is a *pre-Hilbert* seminorm if $\widetilde{E}_p$ is a Hilbert space. If $q \leq p$ is another seminorm on E, the canonical operator from E_p to E_q is denoted by T_{pq}. By $\widetilde{T}_{pq}$: $\widetilde{E}_p \to \widetilde{E}_q$ we denote the canonical extension of T_{pq}.

Proof of the Theorem: Let G be a nuclear group. Due to Theorem (9.6) of [1], there exist a nuclear vector group F, a subgroup P of F and a closed subgroup K of P such that G is topologically isomorphic to P/K. Naturally, we may identify P/K with a subgroup of F/K. As the property of respecting compactness is evidently inherited by arbitrary subgroups, we may simply assume that $G = F/K$. Let β: $F \to G$ be the canonical projection.

Let X be a weakly compact subset of G. First we shall prove that X is totally bounded. Suppose the contrary. Then we can find some $V \in \mathcal{N}_0(G)$ and some sequence $(g_n)_{n=1}^{\infty}$ in X such that $g_m - g_n \notin V$ whenever $m \neq n$. To obtain a contradiction, we shall construct a subsequence of (g_n) without weak cluster points in G.

Choose $U \in \mathcal{N}_0(F)$ such that $\beta(U) \subset V$. By (9.3) and (2.14) of [1], we can find a linear subspace E of F and pre-Hilbert seminorms $p \geq q \geq r$ on E such that $B_r \subset U$, $B_p \in \mathcal{N}_0(F)$, $\sum_{k=1}^{\infty} k\, d_k(\widetilde{T}_{pq}) \leq 1$ and $d_k(\widetilde{T}_{qr}) \to 0$ as $k \to \infty$. We have the canonical commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E \\ \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_r} \\ E_p & \xrightarrow{T_{pq}} & E_q & \xrightarrow{T_{qr}} & E_r \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} \\ \widetilde{E}_p & \xrightarrow{\widetilde{T}_{pq}} & \widetilde{E}_q & \xrightarrow{\widetilde{T}_{qr}} & \widetilde{E}_r \end{array}$$

Set $H = E \cap K$ and consider the canonical commutative diagram

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha"] & F \arrow[d, "\beta"] \\ E/H \arrow[r, "\mu"] & F/K \end{tikzcd}$$

Since $B_p \in \mathcal{N}_0(F)$, the subspace E spanned over B_p is an open subgroup of F, and $A := \beta(E)$ is an open subgroup of $G = F/K$. Observe that μ is a topological embedding. The canonical projection $\gamma\colon G \to G/A$ is continuous if both G and G/A are endowed with their weak topologies, hence $\gamma(X)$ is a weakly compact subset of G/A. As G/A is discrete, Glicksberg's theorem implies that $\gamma(X)$ is compact, hence finite. Therefore, we can choose a subsequence $(g_n')_{n=1}^{\infty}$ of (g_n) such that $\gamma(g_n')$ is constant. Consequently, we can find a sequence $(u_n)_{n=1}^{\infty}$ in E such that $g_n' = \beta(u_n) + g_1'$ for all n.

According to our definitions, we have

$$d(\tilde{T}_{qr}\tilde{T}_{pq}(\psi_p(u_m) - \psi_p(u_n)),\ \tilde{T}_{qr}\tilde{T}_{pq}(\psi_p(K))) \geq 1$$

whenever $m \neq n$. Then it easily follows from Lemma 3 that we can choose a subsequence $(u_{n_s})_{s=1}^{\infty}$ of (u_n) such that the sequence $(\alpha(u_{n_s}))_{s=1}^{\infty}$ does not have any weak cluster points in E/H. In other words, the sequence $(\beta(u_{n_s}))_{s=1}^{\infty}$ does not have weak cluster points in $A = \beta(E) = \mu(E/H)$. Being an open subgroup, A is a weakly closed subset of G. Thus, $(\beta(u_{n_s}) + g_1')_{s=1}^{\infty}$ is a subsequence of (g_n) without weak cluster points in G.

Let us identify G with a subgroup of the completion $\tilde{G}$. Let $\overline{X}$ be the closure of X in $\tilde{G}$. As X is weakly compact, it is weakly closed in $\tilde{G}$, which means that $\overline{X} = X$. Then X is compact, being a closed and totally bounded subset of the complete group $\tilde{G}$. ❑

REMARK 1: A nuclear vector group is not necessarily a topological vector space (cf. [1, p.86]). If F above were indeed a topological vector space, then $E = F$ and the proof would be simpler.

REMARK 2: Following [7], let us denote by $\mathfrak{K}$ and $\mathfrak{P}$ the classes of Abelian topological groups which respect compactness and satisfy Pontryagin duality, respectively. Let A be an open subgroup of an Abelian topological group G. It was observed in [7, Proposition 2.7], that if G belongs to $\mathfrak{K}$ (respectively, to $\mathfrak{P}$), then so does A. The converse is also true: an easy argument shows that $A \in \mathfrak{K} \Rightarrow G \in \mathfrak{K}$, while $A \in \mathfrak{P} \Rightarrow G \in \mathfrak{P}$ was proved in [5, (2.3)].

REMARK 3: It was asked in [7] if every group in $\mathfrak{P} \cap \mathfrak{K}$ can be embedded into a product of LCA groups. The answer is negative; this result had been announced in [8]. Here we give another argument. Corollary 1.5 of [7] says that all Montel spaces belong to $\mathfrak{P} \cap \mathfrak{K}$. On the other hand, it easily follows from the structure theorem for LCA groups that if a topological vector space E can be embedded into a product of LCA groups, then it can be embedded into a product of real lines. So, if E is infinite-dimensional, then every neighborhood of 0 in E contains an infinite dimensional linear subspace. Therefore, for instance, the classical Montel spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{H}$, $\mathcal{S}$ (see, e.g, [10, Section 8, Chapter III]) cannot be embedded into products of LCA groups.

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A Bitopological View on Weight and Cardinality in Almost Stably Locally Compact Spaces

LAWRENCE M. BROWN[a,c] AND MURAT DİKER[b,d]

[a]Hacettepe University
Department of Mathematics, 06532 Beytepe,
Ankara, Turkey
and
[b]Hacettepe University
Department of Science Education, 06532 Beytepe
Ankara, Turkey

ABSTRACT: The class of almost stably locally compact topological spaces is defined and a bitopological characterization obtained. Bounds for the biweight and cardinality of bitopological spaces involved in this characterization are established and applied to give bounds for the weight and cardinality of almost stably locally compact spaces.

1. A BITOPOLOGICAL CHARACTERIZATION

Our aim in this paper is to use a bitopological characterization of a class of stably compactlike spaces to obtain inequalities for the weight and cardinality of spaces in this class. Our characterization is essentially similar to that given in [3] for the class of stably compact spaces itself (called stably locally compact in [3], [8] and in the preprint version of [17]). For the benefit of the reader who may not have access to [3] we shall repeat all relevant definitions and results, outlining the proofs where appropriate.

Recall that a sober topological space (X, u) is called *stably compact* if it is

(a) compact,
(b) locally compact and
(c) the intersection of any two compact saturated subsets is compact.

Here compact does not imply any separation axiom, and locally compact means that every point has a base of compact neighborhoods. A set is *saturated* if it is an upper set for the *specialization order* $x \leq y \Leftrightarrow \overline{x}^u \subseteq \overline{y}^u$, where for $A \subseteq X$, $\overline{A}^u$ denotes the closure of A for the topology u. If we remove the assumption of compactness we have the definition of *stable local compactness* in the sense of [17].

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[c]E-mail: brown@eti.cc.hun.edu.tr.
[d]E-mail: mdiker@eti.cc.hun.edu.tr.

By weakening the sobriety condition slightly we arrive at the class of spaces we wish to study. Recall that (X, u) is *sober* if every irreducible closed set is the closure of a unique singleton. A sober space is therefore necessarily T_0. We may now give:

DEFINITION 1.1: The T_0 space (X, u) is *almost sober* if every irreducible proper closed subset of X is the closure of a unique singleton. An almost sober space satisfying conditions (b) and (c) above is called an *almost stably locally compact* space.

If s (respectively, t) denotes the lower (upper) topology on $\mathbb{R}$ then $(\mathbb{R}, s)$ and $(\mathbb{R}, t)$ are important examples of almost stably locally compact spaces which are not stably locally compact.

We shall find the following result useful in the sequel.

LEMMA 1.1: Let (X, u) be almost stably locally compact and let $C = \{F \cap K\colon F \subset X$ is u-closed and $K \subseteq X$ is u-compact and u-saturated $\}$. Then any nonempty subset $\mathcal{D}$ of C with the finite intersection property has a nonempty intersection.

In case (X, u) is sober we may replace $F \subset X$ in the definition of C by $F \subseteq X$ (compare [3]).

Proof: By Zorn's Lemma there exists a filter $\mathcal{H}$ containing $\mathcal{D}$ which is maximal with respect to the property of having a base contained in C. Define

$$M = \bigcap\{\overline{H}^u\colon H \in \mathcal{H}\}.$$

Since $\mathcal{H}$ has a base of u-compact sets it is trivial to verify that every set in $\mathcal{H}$ meets M. In particular M is a nonempty u-closed set. Let us show that it is irreducible. Suppose on the contrary that $M = A \cup B$, where A, B are u-closed subsets of X with $\varnothing \neq A \subset M$ and $\varnothing \neq B \subset M$. Choose $a \in A\setminus B$ and $b \in B\setminus A$. Then since (X, u) is locally compact we may choose u-open sets R, S and u-compact u-saturated sets P, Q with $a \in R \subseteq P \subseteq X\setminus B$ and $b \in S \subseteq Q \subseteq X\setminus A$. Since $a \in M$ we have $P \cap H \neq \varnothing$ for all $H \in \mathcal{H}$, whence $P \in \mathcal{H}$ by the maximality of $\mathcal{H}$. Likewise $Q \in \mathcal{H}$. However, $P \cap Q \cap M = \varnothing$, which is not possible since $P \cap Q \in \mathcal{H}$, and we have established that M is irreducible.

Since (X, u) is almost sober and we clearly have $M \subset X$ there is a (unique) $x \in M$ with $M = \overline{x}^u$. Take $F \cap K \in \mathcal{D}$. Clearly, $x \in F$ since $F \in \mathcal{H}$ and F is u-closed. On the other hand $K \in \mathcal{H}$ whence $M \cap K = \overline{x}^u \cap K \neq \varnothing$ and $x \in K$ since K is u-saturated. Thus $x \in \bigcap\mathcal{D} \neq \varnothing$.

The restriction $F \subset X$ in the definition of C is only needed to ensure that $M \subset X$, so this may be omitted if (X, u) is sober. ❑

Before stating our characterization theorem we give some background material which may be of particular interest to those not well aquainted with the theory of bitopological spaces. We begin by recalling a method of defining a new topology on X presented in [3].

DEFINITION 1.2: Let S be a family of subsets of X containing X and $\varnothing$ and closed under finite intersections. Define

$$u(S) = \{V \in X\colon x \in V \Rightarrow \exists S \in S,\ x^u \subseteq S \subseteq \overline{S}^u \subseteq V\}.$$

Clearly $u(S)$ is a topology on X. We shall say that S:

(a) has the *base property* if $S' = \{X\backslash S : S \in S\}$ contains a base of u-nhds of x for each $x \in X$;

(b) has the (*weak*) *interpolation property* for u if for every nonempty u-closed set F (every $F = \bar{x}^u$, $x \in X$) and $S \in S$ with $F \subseteq S$ we have $T \in S$ with $F \subseteq T \subseteq \bar{T}^u \subseteq S$;

(c) is *stable* with respect to u if whenever a u-closed proper subset of X is covered by a family of sets in S it is covered by a finite subfamily of these sets;

(d) is ws★ for u if $x \in S \in S \Rightarrow \bar{x}^u \subseteq S$.

Let us relate these properties of S with properties of the bitopological space $(X, u, u(S))$. We shall adopt Kopperman's terminology in [11] for bitopological seperation properties. In particular (X, u, v) is:

(i) *weakly symmetric* (ws) if $x \notin \bar{y}^u \Rightarrow y \notin \bar{x}^v$;

(ii) *pseudo-Hausdorff* (pH) if $x \notin \bar{y}^u \Rightarrow \exists U \in u$, $V \in v$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$;

(iii) *regular* if given $x \in U \in u$ we have $G \in u$ with $x \in G \subseteq \bar{G}^v \subseteq U$;

(iv) *completely regular* if whenever $x \in U \in u$ there is a pairwise continuous function f mapping (X, u, v) into the bitopological unit interval with $f(x) = 1$ and $f(y) = 0$ for all $y \notin U$;

(v) *normal* if whenever $U \in u$, K is v-closed and $K \subseteq U$ we have $G \in u$ with $K \subseteq G \subseteq \bar{G}^v \subseteq U$.

Also T_0 means that $x \in \bar{y}^u \cap \bar{y}^v$ and $y \in \bar{x}^u \cap \bar{x}^v$ implies $x = y$; T_1 is T_0 plus ws; Hausdorff or T_2 is T_0 plus pH; T_3 is T_0 plus regular and T_4 is T_0 plus normal. If Q is any of these properties, Q★ is the same property with the roles of u and v interchanged, while *pairwise* Q means both Q and Q★. Thus regular, completely regular, normal and their pairwise forms are essentially as defined in [9], pairwise ws is the pairwise R_0 of [13], pairwise pH is pairwise R_1 in the sense of [14] (called preseparated in [2]) T_0 (= pairwise T_0) is the weakly pairwise T_0 of [16] and pairwise T_1, pairwise T_2 have been named semipairwise T_1, semipairwise T_2, respectively, by the first author [3].

Let us first note:

LEMMA 1.2: If S has the base property then $(X, u, u(S))$ is pairwise pH. If in addition (X, u) is T_0 then $(X, u, u(S))$ is pairwise T_2.

Proof: It is clear from the definition that $(X, u, u(S))$ is ws★ for any S. We show that when S has the base property $(X, u, u(S))$ is pH, whence pairwise pH follows from [11, Lemma 2.5(b)]. Take $x \notin \bar{y}^u$, so we have $x \in U \in u$, $x \notin U$. By the base property we have $S_0 \in S$ and $G \in u$ with $x \in G \subseteq X\backslash S_0 \subseteq U$. Defining

$$V = \{ z : \bar{z}^u \subseteq S \subseteq X\backslash G \text{ for some } S \in S \}$$

we obtain $y \in V \in u(S)$, while $G \cap V = \varnothing$. If (X, u) is T_0 then $(X, u, u(S))$ is T_0 and hence pairwise T_2 by the above and [11, Lemma 2.5(c)]. ❑

To obtain stronger separation properties we shall make the assumption that S is ws★ for u. Note that if $t(S)$ denotes the topology with S as a base of open sets then it is precisely under this condition that $(X, u, t(S))$ is ws★. We now have:

LEMMA 1.3: Suppose that S is ws★ for u. Then we have the following.

(a) If S has the weak interpolation property we have $u(S) = t(S)$, and $(X, u, u(S))$ is regular★. Additional imposition of the base property gives pairwise regularity.

(b) If S has the interpolation property then $(X, u, u(S))$ is completely regular★. Additional imposition of the base property gives pairwise complete regularity.

(c) If S is stable for u and has the weak interpolation property then $(X, u, u(S))$ is pairwise normal.

Proof: (a) That $u(S) = t(S)$ under the weak interpolation property is evident from the definitions. If $x \in V \in u(S)$ then $\overline{x}^u \subseteq S \subseteq S^u \subseteq V$ for some $S \in S$. Hence, $x \in S \subseteq S^u \subseteq V$ and $S \in t(S) = u(S)$ so $(X, u, u(S))$ is regular★. Suppose now that S also has the base property and take $x \in U \in u$. Then we have $G \in u$ and $S \in S$ with $x \in G \subseteq X \setminus S \subseteq U$. But $X \setminus S$ is $t(S) = u(S)$-closed, so $x \in G \subseteq G^{u(S)} \subseteq U$, i.e., $(X, u, u(S))$ is also regular.

(b) Let $\mathcal{P} = \{ (F, S) : F$ is u-closed, $S \in S$ and $F \subseteq S \}$. By the interpolation property for S it is easy to see that $\mathcal{P}$ is a Urysohn set [11, Definition 2.6], while $\mathcal{T}_{\mathcal{P}} = u(S)$. Thus $(X, u(S), u)$ is completely regular, i.e. $(X, u, u(S))$ is completely regular★, by [11, Lemma 2.7(e)]. In the presence of the base property we also have $\mathcal{T}_{\mathcal{P}\star} = u$, so by the same lemma $(X, u, u(S))$ is pairwise completely regular.

(c) Straightforward. ❑

In the sequel we shall consider the case where S is the set of complements of saturated compact subsets of X. In this case $u(S)$ is the cocompact topology of u [6], [12], or what Kopperman [11] calls the de Groot dual. We have:

LEMMA 1.4: Let $S = \{X \setminus K : K$ is u-compact and u-saturated$\}$. Then:

(a) S is ws★ for u;

(b) if (X, u) is locally compact, S has the base and interpolation properties;

(c) if (X, u) is almost stably locally compact, S is stable.

Proof: (a) and (b) are straightforward. To prove (c) let F be a u-closed proper subset of X and let K_α, $\alpha \in A$, be compact saturated sets with $F \subseteq \bigcup\{ X \setminus K_\alpha : \alpha \in A\}$. If $F \not\subseteq \bigcup\{X \setminus K_\alpha : \alpha \in A'\}$ for all finite subsets A' of A then the family $F \cap K_\alpha$, $\alpha \in A$, has the finite intersection property, and we obtain a contradiction from Lemma 1.1. ❑

We now turn to the question of bitopological compactness. Kopperman calls a bitopological space (X, u, v) *stable* if every v-closed proper subset of X is u-compact. Hence, pairwise stability is the pairwise compactness property introduced in [15], and is characterized by the condition that every *pairwise open cover* of X, i.e., every cover of X by sets in $u \cup v$ which contains at least one nonempty u-open and at least one nonempty v-open set, has a finite subcover. On the other hand (X, u, v) is called *compact* in [11] if (X, u) is compact. Hence, Kopperman's pairwise compactness together with pairwise stability gives *joint compactness*, i.e., compactness with respect to the joint topology $u \vee v$ (called the *specialization topology* $\mathcal{T}^S$ in [11]). The space $(\mathbb{R}, s, t)$ is an example of a pairwise stable space which is not jointly compact. Kopperman also uses the term *joincompact* for a pairwise T_2 jointly compact space.

We may now give the promised characterization:

THEOREM 1.1:

(1) Let (X, u, v) be a pairwise T_2 pairwise stable space. Then (X, u) is an almost stably locally compact space.

(2) Let (X, u) be an almost stably locally compact space. Then there exists a topology v on X so that (X, u, v) is pairwise T_2 and pairwise stable. Moreover v is unique, being equal to the cocompact topology of u.

Stably locally compact spaces are characterized by the additional requirement that (X, u, v) be compact★, while a characterization of stably compact spaces is obtained by replacing pairwise stability by joint compactness (see [3]).

Proof: (1) Clearly (X, u) is T_0 so to prove almost sobriety we need only show that for an u-irreducible u-closed set $M \subset X$ we have $x \in M$ with $M = \overline{x}^u$. Suppose this is not so, then for each $x \in M$ $\exists x' \in M \setminus \overline{x}^u$. Since (X, u, v) is pairwise pH we have $x \in V_x \in v$, $x' \in U_x \in u$ with $U_x \cap V_x = \varnothing$, whence $M \subseteq \bigcup\{V_x : x \in M'\}$ for some finite subset M' of M. Since $V_x \subseteq X \setminus U_x$ for each x we may choose a finite subset $M' = \{x_1, \ldots, x_n\}$ of M with the smallest possible cardinality for which $M \subseteq \bigcup\{X \setminus U_{x_1}, \ldots, X \setminus U_{x_n}\}$. Clearly $n \geq 2$, and it is easy to verify that $M_1 = M \setminus U_{x_1}$, $M_2 = \bigcup_{i=2}^{n}(M \setminus U_{x_i})$ are closed sets with $\varnothing \neq M_k \subset M$, $k = 1, 2$, $M = M_1 \cup M_2$, which contradicts the irreducibility of M.

Note that if (X, v) is compact then the above argument also applies if $M = X$ is u-irreducible, so (X, u) is sober in this case.

Under the given conditions (X, u, v) is pairwise regular. If $x \in U \in u$ has $U \subset X$ then U contains a v-closed and hence u-compact nhd of x. On the other hand if X is the only u-open set containing x then X is a u-compact nhd of x in X. Thus (X, u) is locally compact.

It remains to verify condition (c) of Definition 1.1. For $A \subseteq X$ define $P_v(A) = \{x : x \in \overline{y}^v \text{ for some } y \in A\}$. Then, since (X, u, v) is pairwise pH, we have

(A) A is u-saturated iff $P_v(A) = A$,

(B) A is u-compact iff $P_v(A) = \overline{A}^v$.

It follows that every saturated u-compact set is v-closed, whence (c) merely expresses the fact that the intersection of two v-closed proper subsets is a v-closed proper subset, and hence u-compact.

(2) We first establish the uniqueness of v by showing that if (X, u, v) is pairwise T_2 and pairwise stable then $v = u(S)$, where $S = \{X \setminus K : K \text{ is } u\text{-saturated and } u\text{-compact}\}$. Take $x \in V \in v$. Then we have $V_1 \in v$ with $x \in V_1 \subseteq \overline{V_1}^u \subseteq V$. If we set $K = X \setminus V_1$ then $K \subset X$ is v-closed and hence u-compact, while by (A) and (B) above it is also u-saturated. Hence $V \in u(S)$ so $v \subseteq u(S)$. The reverse inclusion $u(S) \subseteq v$ is trivial, and we have shown that $v = u(S)$.

Now let (X, u) be almost stably locally compact and consider $v = u(S)$ for the family S defined above. By Lemmas 1.2 and 1.3, (X, u, v) is pairwise T_2 and indeed pairwise completely regular and pairwise normal.

To establish pairwise stablity take $F \subset X$ closed in (X, u), and suppose that $\{V_\alpha : \alpha \in A\}$ is a v-open cover of F which has no finite subcover. For $\alpha \in A$ and $x \in V_\alpha$ choose a u-compact u-saturated set $K_\alpha(x)$ with

$$\bar{x}^u \subseteq X\backslash K_\alpha(x) \subseteq \overline{X\backslash K_\alpha(x)}^u \subseteq V_\alpha.$$

Then clearly

$$\mathcal{D} = \{F \cap K_\alpha(x) \colon \alpha \in A,\ x \in V_\alpha\}$$

is a subfamily of C with the finite intersection property, so by Lemma 1.1 we have some $z \in \bigcap \mathcal{D}$. Since $z \in F$ we have $\beta \in A$ with $z \in V_\beta$, whence $z \in K_\beta(z)$, which is clearly impossible. Thus F is v-compact. If (X, u) is sober then the same argument shows that X is also v-compact in this case.

A similar argument may be used to show that every v-closed proper subset of X is u-compact, and the proof is complete. ❑

Note that almost stably local compactness and stable compactness exhibit a basic symmetry which is not shared by stable local compactness.

In the next two sections we will obtain inequalities for the biweight and cardinality of pairwise T_2 pairwise stable bitopological spaces and use the above characterization to deduce corresponding results for the class of almost stably locally compact topological spaces. Cardinal functions in bitopological spaces were first considered by Kopperman and Meyer [10] and by the second author [4], [5].

2. SOME RESULTS ON THE BIWEIGHT AND WEIGHT

Throughout (X, u, v) will be a bitopological space. $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$ is a *bibase* if $\mathcal{B}_u(\mathcal{B}_v)$ is a base for the u-open (v-open) sets. $\mathrm{bw}(X) = \min\{\,|\mathcal{B}| \colon \mathcal{B}$ is a bibase$\}$ is the *biweight* [4], [10] of (X, u, v). Clearly,

$$\mathrm{jw}(X) \le \mathrm{bw}(X) = \max\{\mathrm{w}_u(X), \mathrm{w}_v(X)\},$$

where $\mathrm{jw}(X)$ is the weight of $(X, u \vee v)$. If (X, u, v) is jointly compact and pairwise T_2 then $(X, u \vee v)$ is a compact Hausdorff space so $\mathrm{jw}(X) \le |X|$. We wish to strengthen this result, and will find it convenient to make the following definitions:

DEFINITION 2.1: (a) Let (X, u, v) be pairwise pH, so that for all $x \in X$ we have $\bar{x}^u = \bigcap\{\bar{V}^u \colon x \in V \in v\}$, $\bar{x}^v = \bigcap\{\bar{U}^v \colon x \in U \in u\}$. Then $\mathcal{P} = \mathcal{P}_u \times \mathcal{P}_v \subseteq u \times v$ is a *regular pseudobase* if $\bar{x}^u = \bigcap\{\bar{Q}^u \colon x \in Q \in \mathcal{P}_v\}$, $\bar{x}^v = \bigcap\{\bar{P}^v \colon x \in P \in \mathcal{P}_u\}$ for all $x \in X$, and the *regular pseudoweight* $\mathrm{rpw}(X)$ of X is given by

$$\mathrm{rpw}(X) = \min\{\,|\mathcal{P}| \colon \mathcal{P} \text{ is a regular pseudobase}\}.$$

(b) Let (X, u, v) be a bitopological space, $\mathcal{N}_u$ a network for (X, u) and $\mathcal{N}_v$ a network for (X, v). Then $\mathcal{N} = \mathcal{N}_u \times \mathcal{N}_v$ is a *binetwork* for (X, u, v) and

$$\mathrm{bnw}(X) = \min\{\,|\mathcal{N}| \colon \mathcal{N} \text{ is a binetwork}\}$$

is the *binetwork weight* of (X, u, v).

Note that, since we shall only be interested in the case where the cardinal functions we consider are infinite, we may assume without loss of generality that in (a) the sets $\mathcal{P}_u$ and $\mathcal{P}_v$ cover X, for in the contrary case we need only add the set X to them. We now have:

THEOREM 2.1: Let (X,u,v) be pairwise stable and pairwise T_2. Then

$$\mathrm{bw}(X) = \mathrm{rpw}(X) = \mathrm{bnw}(X) \leq |X|.$$

Proof: To establish $\mathrm{bw}(X) \leq \mathrm{rpw}(X)$ let $\mathcal{P}$ be a regular pseudobase with $|\mathcal{P}| = \mathrm{rpw}(X)$ and define

$$\mathcal{B} = \{(\bigcap_{i=1}^n P_i, \ \bigcap_{i=1}^n Q_i) : (P_i, \ Q_i) \in \mathcal{P}, \ i = 1, 2, \ldots, n; \ n < \omega\}.$$

Certainly $|\mathcal{B}| \leq |\mathcal{P}|$. We claim that $\mathcal{B}$ is a bibase for (X,u,v). Take $G \in u$, $H \in v$, $x \in G \cap H$. Suppose $G \neq X$. Since $\bar{x}^v \subseteq G$, for $y \notin G$ we have $P_y \in \mathcal{P}_u$ with $x \in P_y$, $y \notin \bar{P}_y^{\,v}$. Hence, $\{G\} \cup \{X \backslash \bar{P}_y^{\,v} : y \in X \backslash G\}$ is a pairwise open cover of X, whence for some $y_1, \ldots, y_m \in X\backslash G$ we have $G \cup \bigcup_{i=1}^m (X \backslash \bar{P}_{y_i}^v) = X$, i.e., $\bar{P}_{y_1}^v \cap \ldots \cap \bar{P}_{y_m}^v \subseteq G$. In case $G = X$ this may be satisfied with $m = 1$ and arbitrary $P_{y_1} \in \mathcal{P}_u$ containing x. Likewise we have $Q_{y_{m+1}}, \ldots, Q_{y_n} \in \mathcal{P}_v$ containing x for which $\bar{Q}^u_{y_{m+1}} \cap \ldots \cap \bar{Q}^u_n \subseteq H$. Taking (P_i, Q_i) to be (P_{y_i}, Q_{y_1}) for $1 \leq i \leq m$, and to be (P_{y_1}, Q_{y_i}) for $m+1 \leq i \leq n$ we have $(P_i, \ Q_i) \in \mathcal{P}_u \times \mathcal{P}_v = \mathcal{P}$, $x \in \bigcap_{i=1}^n P_i \subseteq G$ and $x \in \bigcap_{i=1}^n Q_i \subseteq H$, so establishing our claim. Hence $\mathrm{bw}(X) \leq \mathrm{rpw}(X)$ as required.

To verify $\mathrm{rpw}(X) \leq \mathrm{bnw}(X)$, let $\mathcal{N}$ be a binetwork with $|\mathcal{N}| = \mathrm{bnw}(X)$, and let $\mathcal{N}^\star = \{(M, \ N) : (M,N) \in \mathcal{N}, \exists\, (U, V) \in u \times v \text{ with } M \subseteq U, \ N \subseteq V \text{ and } U \cap V = \varnothing\}$. For each $(M, N) \in \mathcal{N}^\star$ choose $U = U_{(M,N)} \in u$, $V = V_{(M,N)} \in v$ with $M \subseteq U$, $N \subseteq V$ and $U \cap V = \varnothing$ and define

$$\mathcal{P}_u = \{U_{(M,N)} : (M,N) \in \mathcal{N}^\star\}, \ \mathcal{P}_v = \{ V_{(M,N)} : (M,N) \in \mathcal{N}^\star\}.$$

Clearly $\mathcal{P} = \mathcal{P}_u \times \mathcal{P}_v$ satisfies $|\mathcal{P}| \leq |\mathcal{N}|$, so it remains to verify that it is a regular pseudobase. However, for $y \notin \bar{x}^u$ we have $U \in u$, $V \in v$ with $y \in U$, $x \in V$ and $U \cap V = \varnothing$. Thus, if we choose $M \in \mathcal{N}_u$, $N \in \mathcal{N}_v$ with $y \in M \subseteq U$ and $x \in N \subseteq V$ we have $(M, N) \in \mathcal{N}^\star$, and then $x \in V_{(M,N)}$, $y \notin U_{(M,N)}$. The case $y \notin \bar{x}^v$ is dealt with in the same way, so $\mathcal{P}$ is a regular pseudobase and we have established $\mathrm{rpw}(X) \leq \mathrm{bnw}(X)$.

The equality of the three cardinal functions now follows from the trivial inequality $\mathrm{bnw}(X) \leq \mathrm{bw}(X)$, while $\mathrm{bnw}(X) \leq |X|$ is immediate from the definition. This completes the proof of the theorem. ❑

The following example shows that the pairwise T_2 condition in Theorem 2.1 cannot be replaced by the condition that each topology be T_1.

EXAMPLE 2.1: Let $X = \mathbb{N} \times \mathbb{N}$, take u to be the topology consisting of $\varnothing$ and all subsets G of X for which the set $\{i : (i, j) \notin G\}$ is finite for all $j \in \mathbb{N}$, and v to be the cofinite topology on X.

Clearly (X, u) is not 2nd countable and so $\mathrm{bw}(X) > |X|$. On the other hand (X,u,v) is clearly pairwise stable and each topology is T_1.

The following result is an immediate corollary to Theorems 1.1 and 2.1:

THEOREM 2.2: Let (X, u) be an almost stably locally compact space. Then $\mathrm{w}(X) \leq |X|$.

It will be interesting to compare the weight $\mathrm{w}_u(X)$ of (X,u) with the weight $\mathrm{w}_v(X)$ of (X,v) under the conditions of Theorem 2.1. That they are in fact equal will emerge as a corollary to the following more general results.

THEOREM 2.3: Let (X,u) be a topological space and define v by $v = u(S)$ as in Definition 1.2. Let

$$S_1 = \min\{|S'|: S' \subseteq S,\ \bar{x} \subseteq T \subseteq \bar{T} \subseteq S \text{ for } S, T \in S \Rightarrow \exists S' \in S', \bar{x} \subseteq S' \subseteq \bar{S}' \subseteq S\},$$

$$S_2 = \min\{|S'|: S' \subseteq S,\ F \subseteq T \subseteq \bar{T} \subseteq S \text{ for } S,T \in S,\ F \text{ closed} \Rightarrow \exists S' \in S',\ F \subseteq S' \subseteq \bar{S}' \subseteq S\},$$

$$S_3 = \min\{|\mathcal{B}|: \mathcal{B} \subseteq u,\ X\backslash S \subseteq G, S \in S,\ G \in u \Rightarrow \exists B \in \mathcal{B}, X\backslash S \subseteq B \subseteq G\}.$$

Then:

(1) if S has the weak interpolation property then $w_v(X) \leq S_1(X) \leq S_2(X)$;
(2) if S has the interpolation property then $S_2(X) \leq S_3(X)$;
(3) if $X\backslash S$ is compact for each $S \in S$ then $S_3(X) \leq w_u(X)$;
(4) if S has the base and interpolation property then $\chi_u(X) \leq S_2(X)$.

Proof: Straightforward. ❑

COROLLARY 1: Under the conditions of Theorem 2.1 we have $w_u(X) = w_v(X)$.

Proof: $w_v(X) \leq w_u(X)$ follows from Theorems 2.1 and 1.1 by noting that $v = u(S)$ where $S = \{X\backslash K: K \text{ compact and saturated}\}$ has the base and interpolation properties. However, by symmetry, we also have $w_u(X) \leq w_v(X)$. ❑

For the above choice of S we obtain:

COROLLARY 2: Let X be an almost stably locally compact space and S the set of complements of compact saturated subsets of X. Then

$$\chi(X) \leq S_1(X) = S_2(X) = S_3(X) = w(X) \leq |X|.$$

In general it is not the case that under the conditions of Theorem 2.1 we have equality for the character, as the following examples show.

EXAMPLE 2.2: Let (X, u) be a discrete, infinite topological space and v the cofinite topology. Then (X,u,v) is pairwise stable and pairwise T_2, $w_u(X) = w_v(X) = |X|$ but $\chi_u(X) \neq \chi_v(X)$.

EXAMPLE 2.3: Define $X = \mathbb{R} \cup \{x^\star, y^\star\}$, where $x^\star, y^\star$ are not in $\mathbb{R}$, let u have the base

$$\{\{x\}: x \in \mathbb{I}\} \cup \{(\mathbb{Q}\backslash A) \cup \{x^\star\}: |A| < \omega, A \subseteq \mathbb{Q}\} \cup \{\mathbb{I} \cup \{y^\star\}\},$$

and v have the base

$$\{\{x\}: x \in \mathbb{Q}\} \cup \{(\mathbb{I}\backslash B) \cup \{y^\star\}: |B| < \omega, B \subseteq \mathbb{I}\} \cup \{\mathbb{Q} \cup \{x^\star\}\},$$

where $\mathbb{Q}$ is the set of rational and $\mathbb{I}$ the set of irrational numbers. Then it is easy to verify that (X,u,v) is pairwise T_2 and jointly compact. Clearly $w_u(X) = w_v(X) = 2^\omega$ but $\omega = \chi_u(X) < \chi_v(X) = 2^\omega$.

3. BOUNDS ON THE CARDINALITY

If (X, u, v) is jointly compact and pairwise T_2 then $(X, u \vee v)$ is a compact Hausdorff space and $|X| \leq \exp(j\chi(X)) = \exp(j\psi(X))$ by the famous theorem of Arhangel'skiĭ [1]. This is clearly false if we replace joint compactness by pairwise stability as Example 2.2 shows. However, as we now show, it is possible to find an estimate for the cardinality of an even wider class of spaces, namely the pairwise T_1 pairwise stable spaces.

Let (X, u, v) be a pairwise ws space. We say that $\mathcal{B}(x) = \mathcal{B}_u(x) \times \mathcal{B}_v(x)$, where $\mathcal{B}_u(x) \subseteq u$ and $\mathcal{B}_v(x) \subseteq v$, is a *pseudobibase* at x if $\bar{x}^u = \bigcap\{V: V \in \mathcal{B}_v(x)\}$, $\bar{x}^v = \bigcap\{U: U \in \mathcal{B}_u(x)\}$ and define

$$\mathrm{b}\psi(x, X) = \min\{|\mathcal{B}(x)|: \mathcal{B}(x) \text{ is a pseudobibase at } x\}$$

and

$$\mathrm{b}\psi(X) = \sup\{\mathrm{b}\psi(x, X): x \in X\}.$$

For a not necessarily T_1 topological space (X, u) a pseudobase $C(x)$ at x may be defined by the conditions $C(x) \subseteq u$, $\bigcap C(x) = (x)$ where $(x) = \bigcap\{U: x \in U \in u\}$. The pseudocharacter $\psi_u(x, X)$ at x is then the minimum cardinal of such a pseudobase at x, while the pseudocharacter $\psi_u(X)$ of (X, u) is the supremum of $\psi_u(X, x)$ over $x \in X$. We then have:

$$j\psi(X) \leq \mathrm{b}\psi(X) = \max\{\psi_u(X), \psi_v(X)\}.$$

Clearly $\mathrm{b}\psi(X) \leq \mathrm{b}\chi(X)$, where the bicharacter is defined in the obvious way, and we have equality for pairwise stable pairwise pH spaces. We may now state the promised theorem:

THEOREM 3.1: Let (X, u, v) be pairwise stable and pairwise T_1. Then

$$|X| \leq \exp(\mathrm{b}\psi(X)).$$

Our proof will follow the same pattern as A. A Gryzlov's proof of the inequality $|X| \leq \exp(\psi(X))$ for compact T_1 topological spaces [7]. First we shall need a notion of initial pairwise stability for bitopological spaces. Throughout the following, τ is an infinite cardinal.

DEFINITION 3.1: (X, u, v) is *initially pairwise stable up to* τ if every pairwise open cover b of X with $|b| \leq \tau$ has a finite subcover.

Recall [2] that a *bifilter subbase* $\mathcal{B}$ is a product of filter subases, and that $\mathcal{B}$ is called *ι-regular* if $P\mathcal{B}Q \Rightarrow P \cap Q \neq \varnothing$. A point x is a *cluster point* of $\mathcal{B}$ if $x \in \overline{P}^v \cap \overline{Q}^u$ $\forall\, P\mathcal{B}Q$. If now we call $\mathcal{B}$ *essential* if $\exists\, P\mathcal{B}Q$ with $\overline{P}^v \subset X$ and $\overline{Q}^u \subset X$ we may give the following characterization.

LEMMA 3.1: (X, u, v) is initially pairwise stable up to τ iff every ι-regular essential bifilter subbase $\mathcal{B}$ in X with $|\mathcal{B}| \leq \tau$ has a cluster point in X.

Proof: Straightforward. ❑

Clearly a similar characterization of pairwise stability may be given by omitting the cardinality restriction.

For the proof of Gryzlov's theorem the notion of initial compactness up to τ is sufficient, but it transpires that for the bitopological case the corresponding notion of initial pairwise stability up to τ is not sufficiently powerful, and we introduce the following concept.

DEFINITION 3.2: $Y \subseteq X$ has the *$pc(\tau)$-property* if for all infinite subsets B of Y with $|B| \leq \tau$ we have:

(a) $\overline{B}^u \subset X \Rightarrow B$ has a v-total accumulation point (v-t.a.p.) in $\overline{B}^u \cap Y$

(b) $\overline{B}^v \subset X \Rightarrow B$ has a u-total accumulation point (u-t.a.p.) in $\overline{B}^v \cap Y$.

Here we recall that x is a total (or, complete) accumulation point of a subset A of a topological space if for each open nhd. G of x we have $|U \cap G| = |G|$.

The next result shows that this property involves elements of both topological initial compactness and bitopological initial pairwise stability.

LEMMA 3.2: Suppose $Y \subseteq X$ has the $pc(\tau)$-property. Then we have the following.

(1) Let $A \subseteq Y$. If A is u-closed (v-closed) in Y and $\overline{A}^u \subset X$ ($\overline{A}^v \subset X$) then A is initially v-compact (u-compact) up to τ.

(2) Y is initially pairwise stable up to τ.

Proof: (1) Let A be an u-closed subset of Y and suppose $\overline{A}^u \subset X$. Let $\mathcal{V}$ be a family of v-open subsets of X with $A \subseteq \bigcup\mathcal{V}$ and $|\mathcal{V}| \leq \tau$. Suppose that $\mathcal{V}$ has no finite subcover of A, and that $\mathcal{V}$ has the smallest possible cardinal. Equip $\mathcal{V}$ with a minimal well order $\prec$. For each $V \in \mathcal{V}$ we may choose $x(V) \in A$ with $x(V) \in X\backslash\bigcup\{\{x(V')\} \cup V' : V' \prec V \text{ and } V' \neq V\}$. The set $B = \{x(V) : V \in \mathcal{V}\}$ is infinite and $|B| \leq \tau$. Now $B \subseteq A$ and $\overline{A}^u \subset X \Rightarrow \overline{B}^u \subset X$ so B has a v-t.a.p. $x \in \overline{B}^u \cap Y \subseteq \overline{A}^u \cap Y = A$ since A is u-closed in Y. Take $V_x \in \mathcal{V}$ with $x \in V_x$. Then $x(V) \in B \cap V_x \Rightarrow V \prec V_x$ and $V \neq V_x$ so $|B \cap V_x| < |\mathcal{V}| = |B|$ which contradicts the fact that x is a v-t.a.p. of B.

(2) Let $\mathcal{G}$ be a pairwise open (in X) cover of Y with $|\mathcal{G}| \leq \tau$. Let $G = \bigcup\{U : U \in \mathcal{G} \cap u\}$, $H = \bigcup\{V : V \in \mathcal{G} \cap v\}$.

(i) Suppose $Y \subseteq H$. Choose $U_0 \in \mathcal{G} \cap u$ with $U_0 \neq \varnothing$ and let $A = Y \cap (X\backslash U_0)$. Then A is u-closed in Y and $\overline{A}^u \subseteq X\backslash U_0 \subset X$ so by (1) we can find $V_1, \ldots, V_n \in \mathcal{G} \cap v$ with $A \subseteq V_1 \cup \ldots \cup V_n$, whence $\{U_0, V_1, \ldots, V_n\}$ is a finite subcover of Y.

(ii) Suppose $Y \subseteq G$. A similar proof holds.

(iii) Suppose $Y \nsubseteq H$ and $Y \nsubseteq G$. Let $A = Y \cap (X\backslash G)$. As above we have $V_1, \ldots, V_n \in \mathcal{G} \cap v$ with $A \subseteq V_1 \cup \ldots \cup V_n$. Now let $A' = Y \cap X\backslash(\bigcup_{i=1}^n V_i)$ and we can choose $U_1, \ldots, U_m \in \mathcal{G} \cap u$ with $A' \subseteq U_1 \cup \ldots \cup U_m$. Then $\{U_1, \ldots, U_m, V_1, \ldots, V_n\}$ is a finite subcover of Y. ❑

LEMMA 3.3: Let (X, u, v) be pairwise stable and pairwise T_1. Then if $\varnothing \neq Y \subseteq X$ has the $pc(\tau)$-property for $\tau = b\psi(X)$ we have:

(1) Y is pairwise stable.

(2) If Y satisfies $x \notin Y \Rightarrow Y \subseteq \overline{x}^u \subset X$ or $Y \subseteq \overline{x}^v \subset X$ then:

(i) $\exists x \notin Y,\ Y \subseteq \overline{x}^u \Rightarrow Y$ is v-compact;

(ii) $\exists x \notin Y,\ Y \subseteq \overline{x}^v \Rightarrow Y$ is u-compact.

Proof: (1) Let $\mathcal{B}(x)$ be a pseudo dual base at x with $|\mathcal{B}(x)| \leq \tau$. Let $\mathcal{B}$ be a maximal $\imath$-regular closed essential bifilter in Y. Then $\overline{\mathcal{B}} = \{(\overline{P}^v, \overline{Q}^u): P\mathcal{B}Q\}$ is an $\imath$-regular closed essential bifilter base on X, whence $\exists x \in \bigcap\{\overline{P}^v \cap \overline{Q}^u: P\mathcal{B}Q\}$ since (X, u, v) is pairwise stable. We show that if $x \in G \in u$, $x \in H \in v$ then there exists $P\mathcal{B}Q$ with $P \cap Q \subseteq G \cap H$. Suppose not. Then for some G, H we have $P \cap Q \not\subseteq G \cap H$, i.e., $P \cap Q \cap (X\backslash G) \neq \varnothing$ or $P \cap (X\backslash H) \cap Q \neq \varnothing$ for all $P\,\mathcal{B}\,Q$. There are two cases to consider:

(a) Suppose $P \cap Q \cap (X\backslash G) \neq \varnothing$ for all $P\mathcal{B}Q$. Then

$$\{(P,\ Q \cap (X\backslash G)): P\mathcal{B}Q\}$$

is a base for an $\imath$-regular closed essential bifilter $\mathcal{B}'$ on Y. $(P,\ Q \cap (X\backslash G)) \notin \mathcal{B}$ since otherwise $x \in \overline{Q \cap (X\backslash G)}^u$ would give the contradiction $G \cap Q \cap (X\backslash G) \neq \varnothing$. On the other hand $\mathcal{B} \subseteq \mathcal{B}'$ since $P \subseteq P$ and $Q \cap (X\backslash G) \subseteq Q$ for all $P\mathcal{B}Q$. This contradicts the maximality of $\mathcal{B}$.

(b) Suppose $\exists P_0\mathcal{B}Q_0$ with $P_0 \cap Q_0 \cap (X\backslash G) = \varnothing$. Define

$$C = \{(P \cap (X\backslash H),\ Q): P\mathcal{B}Q, P \cap Q \cap (X\backslash G) = \varnothing\}.$$

Then $C \neq \varnothing$ and if $(P_i \cap (X\backslash H), Q_i) \in C$, $i = 1, 2$ then $P_i \cap Q_i \cap (X\backslash G) = \varnothing \Rightarrow P_1 \cap P_2 \cap Q_1 \cap Q_2 \cap (X\backslash G) = \varnothing \Rightarrow (P_1 \cap P_2 \cap (X\backslash H),\ Q_1 \cap Q_2) \in C$. Also $P \cap Q \cap (X\backslash G) = \varnothing \Rightarrow P \cap (X\backslash H) \cap Q \neq \varnothing$ so C is a base for an $\imath$-regular closed essential bifilter $\mathcal{B}'$ on Y. Now $P\mathcal{B}Q \Rightarrow (P \cap P_0 \cap (X\backslash H),\ Q \cap Q_0) \in C$ and $P \cap P_0 \cap (X\backslash H) \subseteq P$, $Q \cap Q_0 \subseteq Q$, so $\mathcal{B} \subseteq \mathcal{B}'$. On the other hand $(P_0 \cap (X\backslash H),\ Q_0) \notin C$ since $x \in \overline{P_0 \cap (X\backslash H)}^v$ and $x \in H \in v$ would give the contradiction $H \cap P_0 \cap (X\backslash H) \neq \varnothing$. This again contradicts the maximality of $\mathcal{B}$.

For each $(G, H) \in \mathcal{B}(x)$ we may therefore choose $P(G, H)\mathcal{B}Q(G, H)$ with $P(G, H) \cap Q(G, H) \subseteq G \cap H$. Without loss of generality we may assume $P(G, H) \subset Y$ and $Q(G, H) \subset Y$ since $\mathcal{B}$ is essential. Thus

$$\mathcal{B}^\star = \{(P(G, H),\ Q(G, H)): (G, H) \in \mathcal{B}(x)\}$$

is an $\imath$-regular closed essential bifilter subbase on Y, and $|\mathcal{B}^\star| \leq \tau$. By Lemma 3.2(1), Y is initially pairwise stable up to τ and so by Lemma 3.1, $\mathcal{B}(x)$ has a cluster point in Y. Thus $y \in \bigcap\{P(G, H) \cap Q(G, H): (G, H) \in \mathcal{B}(x)\} \subseteq \bigcap\{G \cap H: (G, H) \in \mathcal{B}(x)\} = \overline{x}^v \cap \overline{x}^u = \{x\}$, since (X, u, v) is pairwise T_1. Hence, $y = x$ so $y \in Y \cap \overline{P}^v \cap \overline{Q}^u = P \cap Q$ for all $(P, Q) \in \mathcal{B}$. Thus Y is pairwise stable.

(2)(i) Suppose $Y \subseteq \overline{x_0}^u$ for some $x_0 \in X$. By hypothesis $\overline{x_0}^u \subset X$, so we have $\overline{Y}^u \subset X$. Let $\mathcal{K}$ be a maximal filter of v-closed subsets of Y and assume that $\bigcap\mathcal{K} = \varnothing$. Let

$$\overline{\mathcal{K}} = \{\overline{K}^v \cap \overline{Y}^u: K \in \mathcal{K}\}.$$

Then $\overline{Y}^u$ is v-compact since (X, u, v) is pairwise stable and $\overline{Y}^u \subset X$, so $\exists\, x \in \bigcap\{\overline{K}^v \cap \overline{Y}^u: K \in \mathcal{K}\}$. Clearly $x \notin Y$ otherwise $x \in \bigcap\{\overline{K}^v \cap Y: K \in \mathcal{K}\} = \bigcap\mathcal{K}$ which contradicts $\bigcap\mathcal{K} = \varnothing$. There are two cases to consider:

Case 1: $Y \subseteq \overline{x}^v$. Then for all $K \in \mathcal{K}$, $Y = \overline{x}^v \cap Y \subseteq \overline{K}^v \cap Y = K \subseteq Y$ since K is v-closed in Y. Thus $\mathcal{K} = \{Y\}$ has $\bigcap\mathcal{K} \neq \varnothing$, which is a contradiction.

Case 2: $Y \subseteq \overline{x}^u$. We may choose a system $\mathcal{B}_u(x)$ of u-neighborhoods of x with $|\mathcal{B}_u(x)| \leq \tau$ and $\bigcap\mathcal{B}_u(x) = \overline{x}^v$. Now for each $U \in \mathcal{B}_u(x)$ we have $K_U \in \mathcal{K}$ with $K_U \subseteq U$. For suppose not, then for some U we have $K \not\subseteq U$ for all $K \in \mathcal{K}$, i.e., $(X\backslash U)$

$\cap K \neq \emptyset$ so $(X\backslash U) \cap \overline{K}^v \neq \emptyset$ for all $K \in \mathcal{K}$. Now $X\backslash U \subset X$ is u-closed and hence v-compact and $\{ (X\backslash U) \cap \overline{K}^v: K \in \mathcal{K} \}$ is a filter base of v-closed subsets so $\exists\, z \in (X\backslash U) \cap \bigcap_{K \in \mathcal{K}} \overline{K}^v$. Again if $z \in Y$ we have a contradiction so suppose $z \notin Y$. There are again two cases:

(α) $Y \subseteq \overline{z}^v$. Then $Y \subseteq \overline{z}^v \subseteq \overline{K}^v \Rightarrow Y = \overline{K}^v \cap Y = K$, which gives the contradiction $\mathcal{K} = \{Y\}$.

(β) $Y \subseteq \overline{z}^u$. Then $x \in \overline{Y}^u \subseteq \overline{z}^u$. But $x \in U \in \mathcal{B}(x)$ and $x \notin U$, which again is a contradiction.

This establishes the existence of K_U and the family $\mathcal{K}^\star = \{K_U : U \in \mathcal{B}_u(x)\}$ clearly has $|\mathcal{K}^\star| \leq \tau$, whence $\exists y \in \bigcap\{K_U : U \in \mathcal{B}_u(x)\}$ by Lemma 3.2(1) with $A = Y$. Thus $y \in \bigcap \mathcal{B}_u(x) = \overline{x}^v$. However, $y \in Y \subseteq \overline{x}^u$ which gives the contradiction $x = y \in Y$ as before.

This proves $\bigcap \mathcal{K} \neq \emptyset$, whence Y is v-compact.

(2)(ii) Suppose $Y \subseteq \overline{x_0}^v$ for some $x_0 \in X$. A similar argument establishes that Y is u-compact. ❑

LEMMA 3.4: Let (X, u, v) be pairwise stable and take $Y \subseteq X$. If $\overline{Y}^u \subset X$ then Y has a v-t.a.p. in $\overline{Y}^u$ and if $\overline{Y}^v \subset X$ then Y has a u-t.a.p. in Y.

Proof: Straightforward. ❑

LEMMA 3.5: Let (X, u, v) be pairwise stable, $A \subseteq X$ have $|A| \leq 2^\tau$. Then we have $F \subseteq X$ with the $pc(\tau)$-property satisfying $A \subseteq F$ and $|F| \leq 2^\tau$.

Proof: Construct a family $\{A_\alpha : \alpha \leq \omega_{\tau^+}\}$ of subsets of X having $|A_\alpha| \leq 2^\tau$ by

(1) $A_0 = A$

(2) Suppose $\alpha = \beta + 1$ and let $W_\beta = \{ B : B \subseteq A_\beta, |B| \leq \tau \}$, $W_\beta{}^u = \{B : B \in W_\beta, \overline{B}^u \subset X \}$ and $W_\beta{}^v = \{B : B \in W_\beta, \overline{B}^v \subset X\}$. By Lemma 3.4 each $B \in W_\beta{}^u$ has a v-t.a.p. $x_B{}^u$ in $\overline{B}^u$ and each $B \in W_\beta{}^v$ has a u-t.a.p. $x_B{}^v$ in $\overline{B}^v$. Let

$$A_\alpha = A_\beta \cup \{x_B{}^u : B \in W_\beta{}^u\} \cup \{x_B{}^v : B \in W_\beta{}^v\}.$$

(3) Suppose α is a limit ordinal. In this case let

$$A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}.$$

Clearly $|A_\alpha| \leq 2^\tau$ for each $\alpha < \omega_{\tau^+}$. Hence

$$F = \bigcup\{A_\alpha : \alpha < \omega_{\tau^+}\}$$

has $|F| \leq 2^\tau$, and clearly $A \subseteq F$. It is easy to verify that F has the $pc(\tau)$-property. ❑

Proof of Theorem 3.1: Let $\tau = b\psi(X) = \max(\psi_u(X), \psi_v(X))$, and for each $x \in X$ let $\mathcal{B}_u(x)$, $\mathcal{B}_v(x)$ be families of u, v-neighborhoods of x with $|\mathcal{B}_u(x)| \leq \tau$ and $|\mathcal{B}_v(x)| \leq \tau$. Let γ be a choice function on the nonempty subsets of X. We construct a sequence $\{F_\alpha : \alpha < \omega_{\tau^+}\}$ of subsets of X with the $pc(\tau)$-property and satisfying $|F_\alpha| \leq 2^\tau$ as follows.

(1) Let F_0 be any subset of X of cardinality not exceeding 2^τ which includes any point x with $\overline{x}^u = X$ (necessarily unique since (X, u) is T_0), and any point x with $\overline{x}^v = X$ (necessarily unique since (X, v) is T_0).

(2) Let $\alpha = \beta + 1$. Consider $W_\beta = \bigcup\{ \mathcal{B}_u(x) \cup \mathcal{B}_v(x) : x \in F_\beta \}$ and let

$$F_\alpha' = F_\beta \cup \{\gamma(X\setminus\bigcup\mathcal{K}) : \mathcal{K} \subseteq W_\beta \text{ finite}, \bigcup\mathcal{K} \subset X\}.$$

Clearly $|F_\alpha'| \le 2^\tau$. Let F_α be a set with the $pc(\tau)$ property containing F_α' and satisfying $|F_\alpha| \le 2^\tau$ constructed as in Lemma 3.5 for $A = F_\alpha'$.

(3) Let α be a limit ordinal. In this case let

$$F_\alpha' = \bigcup_{\beta<\alpha} F_\beta$$

and choose F_α as in (2).

Now consider the set $D = \bigcup\{F_\alpha : \alpha < \omega_{\tau^+}\}$. Then $|D| \le 2^\tau$ and clearly D has the $pc(\tau)$-property. Now suppose that $D \subset X$. There are two cases to consider.

(i) For all $x \notin D$ we have $D \subseteq \bar{x}^u$ or $D \subseteq \bar{x}^v$. Suppose that $D \subseteq \bar{x}^u$ for some $x \notin D$. In view of the definition of $F_0 \subseteq D$ we may apply Lemma 3.3(2) and deduce that D is v-compact. Then $y \in D \Rightarrow y \in \bar{x}^u \Rightarrow y \notin \bar{x}^v$ since the space is pairwise T_1. Hence $x \notin \bar{y}^u$ since (X, u, v) is pairwise ws. But $\bar{y}^u = \bigcap\mathcal{B}_v(y)$ so we have $V_y \in \mathcal{B}_v(y)$ with $x \notin V_y$. Thus $D \subseteq V_{y_1} \cup \ldots \cup V_{y_n}$ for some $y_1, \ldots, y_m \in D$. We have $\alpha < \omega_{\tau^+}$ with $y_1, \ldots, y_m \in F_\alpha$ so $\mathcal{K} = \{V_{y_1}, \ldots, V_{y_m}\} \subset W_\alpha$. Thus $\gamma(X\setminus\bigcup\mathcal{K}) \in F_{\alpha+1} \subseteq D$, which is a contradiction. A similar contradiction is also obtained if $D \subseteq \bar{x}^v$ for some $x \notin D$.

(ii) $\exists x \notin D$, $D \not\subseteq \bar{x}^u$ and $D \not\subseteq \bar{x}^v$. Take $y \in D\setminus\bar{x}^u$ and $z \in D\setminus\bar{x}^v$. Then we have $U_y \in \mathcal{B}_u(y)$, $V_z \in \mathcal{B}_v(z)$ with $x \notin U_y \cup V_z$. For $t \in D$, $t \ne y, z$ we have $t \notin \bar{x}^u$ or $t \notin \bar{x}^v$ so we have $L_t \in \mathcal{B}_u(t) \cup \mathcal{B}_v(t)$ with $x \notin L_t$. Hence $\{U_y, V_z, L_t : t \in D\setminus\{y, z\}\}$ is a pairwise open cover of D, and D is pairwise stable by Lemma 3.3(1) so we have $t_1, \ldots, t_n \in D$ for which $\mathcal{K} = \{U_y, V_z, L_{t_1}, \ldots, L_{t_n}\}$ is a cover of D. Again $\mathcal{K} \in W_\alpha$ for some $\alpha < \omega_\tau$ and we have the contradiction $\gamma(X\setminus\bigcup\mathcal{K}) \in F_{\alpha+1} \subseteq D$. This shows that $D = X$ so $|X| = |D| \le 2^\tau = 2^{b\psi(X)}$ as required. ❑

In order to deduce an upper bound for the cardinality of an almost stably locally compact space we must characterize $\psi_v(X)$. For each $x \in X$ the set $X\setminus\bar{x}^u$ may be expressed as a union of compact saturated sets. Let $C(x)$ be a family of compact saturated sets satisfying

$$X\setminus\bar{x}^u = \bigcup C(x)$$

and denote by $\mathrm{sc}\psi(x, X)$ the minimum cardinality of such families. If we let

$$\mathrm{sc}\psi(X) = \sup\{\mathrm{sc}\psi(x, X) : x \in X\}$$

then we have the following lemma.

LEMMA 3.6: If (X, u, v) is pairwise stable and pairwise T_2 then for all $x \in X$ we have $\mathrm{sc}\psi(x, X) = \psi_v(x, X)$ so $\mathrm{sc}\psi(X) = \psi_v(X)$.

Proof: Immediate from properties (A) and (B) in the proof of Theorem 1.1. ❑

We now obtain the following corollary to Theorem 3.1:

THEOREM 3.2: Let (X, u) be almost stably locally compact. Then

$$|X| \le \exp(\psi(X)\mathrm{sc}\psi(X)).$$

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Perfect Maps and Relatively Discrete Collections

DENNIS K. BURKE[a] AND ROGER W. HANSELL[b]

[a]*Department of Mathematics*
Miami University
Oxford, Ohio 45056
and
[b]*Department of Mathematics*
University of Connecticut
Storrs, Connecticut 06269

ABSTRACT: We show that the image of a σ-relatively discrete cover of a space under a perfect map has a σ-relatively discrete refinement. From this we deduce that spaces with a σ-relatively discrete network and, in particular, σ-relatively discrete sets, are invariant under perfect maps. Another corollary is that weakly θ-refinable spaces are preserved by perfect maps, a result previously shown by the first author.

The purpose of this note is to prove several "perfect image theorems" concerning spaces defined in terms of σ-relatively discrete collections. Such spaces arise naturally in connection with the study of general analytic topological spaces [3].

We first recall some definitions. A collection $\mathcal{N}$ of disjoint subsets of a space X is *relatively discrete* if each point of $\bigcup\mathcal{N}$ has a neighborhood that meets only one member of $\mathcal{N}$ (equivalently, there are open sets U_N in X such that $N \subset U_N$ and $U_N \cap N' = \varnothing$ for all $N' \neq N$). Recall that a cover $\mathcal{N}$ of a space X is a *network* for X if, whenever U is an open neighborhood of x, then $x \in N \subset U$ for some $N \in \mathcal{N}$. When we say that $\mathcal{N} = \bigcup_{n\in\omega}\mathcal{N}_n$ is a σ-relatively discrete network for X we mean that $\mathcal{N}$ is a network and $\mathcal{N}_n$ is relatively discrete for each $n \in \omega$. Finally, a collection $\mathcal{N}$ is a *baselike refinement* for a collection $\mathcal{M}$ if $\mathcal{N}$ is a refinement of $\mathcal{M}$ and each member of $\mathcal{M}$ is a union of sets from $\mathcal{N}$.

The proof of our main theorem will make use of the following lemma.

LEMMA: Any locally finite collection has a σ-relatively discrete baselike refinement.

Proof: Let $\mathcal{E} = \{E_a \colon a \in A\}$ be a locally finite collection of subsets of the space X. For each $n = 1, 2, \ldots$ let $\mathcal{P}_n = \{B \subset A \colon \mathrm{card}(B) = n\}$, and for each $B \in \mathcal{P}_n$ define

$$E_B = \{x \in \textstyle\bigcap_{a\in B} E_a : x \in \overline{E}_a \leftrightarrow a \in B\} \text{ and } U_B = X - \textstyle\bigcup\{\overline{E}_a : a \in A - B\}.$$

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Note that $E_B \subset U_B$, U_B is open in X, and $E_{B'} \cap U_B = \emptyset$ for all $B' \in \mathcal{P}_n$ such that $B' \neq B$. Thus $\{E_B : B \in \mathcal{P}_n\}$is relatively discrete for each $n = 1, 2, \ldots$. It follows that $\bigcup_{n=1}^{\infty} \bigcup\{E_B : B \in \mathcal{P}_n\}$is the desired σ-relatively discrete baselike refinement of $\mathcal{E}$.

THEOREM 1: Suppose $f: X \to Y$ is a perfect onto map and $\mathcal{N}$ is a σ-relatively discrete cover for X. Then Y has a σ-relatively discrete cover $\mathcal{M}$ which is a refinement of $\{f(N) : N \in \mathcal{N}\}$. Moreover, if $\mathcal{N}$ is a network for X, then $\mathcal{M}$ is a network for Y.

Proof: Let $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$ be a cover for X (respectively, a network) such that, for each $n \in \omega$, $\mathcal{N}_n$ is relatively discrete, and let $\mathcal{U}_n = \{U_N : N \in \mathcal{N}_n\}$ be a collection of open sets in X such that, whenever $N \neq N'$ in $\mathcal{N}_n$, we have $N \subseteq U_N$ and $N' \cap U_N = \emptyset$.

For each $y \in Y$ we consider the set $\Omega(y)$ of all finite increasing sequences $n_0 < n_1 < \cdots < n_k$ in ω such that

(1) $f^{-1}(y) \subseteq \bigcup_{i\leq k}(\bigcup \mathcal{U}_{n_i})$, and

(2) there exists $N \in \mathcal{N}_{n_k}$ such that $f^{-1}(y) \cap (N \setminus \bigcup_{i<k}(\bigcup \mathcal{U}_{n_i})) \neq \emptyset$.

Put $\Omega = \bigcup \{\Omega(y) : y \in Y$. For each $n|k = (n_0, \ldots, n_k) \in \Omega$, let

$$Y_{n|k} = \{y \in Y : n|k \in \Omega(y)\},$$

$$\mathcal{M}_{n|k} = \{(N \setminus \bigcup_{i<k}(\bigcup \mathcal{U}_{n_i})) \cap f^{-1}(Y_{n|k}) : N \in \mathcal{N}_{n_k}\}.$$

It is easy to see that $\mathcal{M}_{n|k}$ is a discrete collection of subsets of $\bigcup_{i\leq k}(\bigcup \mathcal{U}_{n_i})$ and hence also of the subspace $X_{n|k} = f^{-1}(Y_{n|k})$.

Since f restricted to $X_{n|k}$ is a perfect map onto $Y_{n|k}$, a standard argument shows that the image of the discrete collection $\mathcal{M}_{n|k}$ under this map is locally finite relative to $Y_{n|k}$. Moreover, by the Lemma, any locally finite collection has a σ-relatively discrete base-like refinement. It follows that there is a sequence $\langle \mathcal{L}_n \rangle_{n\in\omega}$ of relatively discrete collections in Y such that, whenever $y \in f(M)$ for some $M \in \mathcal{M}_{n|k}$ and $n|k \in \omega^{k+1}$, then $y \in L \subseteq f(M)$ for some $L \in \mathcal{L}_m$ and $m \in \omega$. It thus suffices to show that the collection

$$\bigcup_{k\in\omega} \bigcup_{n|k\in\omega^{k+1}} \{f(M) : M \in \mathcal{M}_{n|k}\}$$

is a cover (respectively, a network) for Y— since then $\langle \mathcal{L}_n \rangle_{n\in\omega}$ will be the desired σ-relatively discrete refinement of $\{f(N) : N \in \mathcal{N}\}$(respectively, network for Y).

We now give the proof in the case when $\mathcal{N}$ is a network and point out that for the other case one need only take $V = Y$ in the following argument and substitute the word "cover" for "network." Let V be an open neighborhood of the point $y \in Y$, and let us show there is some $M \in \mathcal{M}_{n|k}$ with $y \in f(M) \subseteq V$. Since $f^{-1}(y) \subseteq f^{-1}(V)$ and $\mathcal{N}$ is a network for X, there is a smallest $n_0 \in \omega$ such that, for some $x \in f^{-1}(y)$, there is an $N \in \mathcal{N}_{n_0}$ such that $x \in N \subseteq f^{-1}(V)$. Note that, if

$$f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_0},$$

then $n_0 \in \Omega(y)$, $N \cap f^{-1}(Y_{n_0}) \in \mathcal{M}_{n_0}$, and $y \in f(N) \cap Y_{n_0} \subseteq V$ as required. Otherwise, there is some $x \in f^{-1}(y) \setminus \bigcup \mathcal{U}_{n_0}$ and so we can find a smallest $n_1 > n_0$ and $N \in$

$\mathcal{N}_{n_1}$ such that $x \in N \subseteq f^{-1}(V)$. Continuing in this way we generate a sequence $\langle n_i \rangle$, where n_i exists provided

$$f^{-1}(y) \setminus \bigcup_{j<i} \left(\bigcup \mathcal{U}_{n_j}\right) \neq \varnothing,$$

in which case some $N \in \mathcal{N}_{n_i}$ must intersect this set and satisfy $N \subseteq f^{-1}(V)$, n_i being the smallest index for which such an N exists. We claim this process must terminate at some finite stage. Otherwise the intersection

$$\bigcap_{i \in \omega}\left(f^{-1}(y) \setminus \bigcup_{j<i} \left(\bigcup \mathcal{U}_{n_j}\right)\right)$$

contains a point x, and for some $n \in \omega$ and for some $N \in \mathcal{N}_n$ we have $x \in N \subseteq f^{-1}(V)$. But this contradicts the minimality of the first $n_i > n$. Hence, for some $k \in \omega$ and for some $N \in \mathcal{N}_{n_k}$ we have $n|k \in \Omega(y)$, $N \subseteq f^{-1}(V)$, and

$$f^{-1}(y) \cap \left(N \setminus \bigcup_{i<k}\left(\bigcup \mathcal{U}_{n_i}\right)\right) \neq \varnothing.$$

Consequently, if

$$M \equiv \left(N \setminus \bigcup_{i<k}\left(\bigcup \mathcal{U}_{n_i}\right)\right) \cap f^{-1}(Y_{n|k}),$$

then $M \in \mathcal{M}_{n|k}$, $M \cap f^{-1}(y) \neq \varnothing$, and thus $y \in f(M) \subseteq V$ as required. That concludes the proof. ❑

Since a space X is σ-relatively discrete if and only if $\{\{x\} : x \in X\}$ is a σ-relatively discrete network for X, the following is an immediate corollary of the theorem.

COROLLARY 1: Let $f : X \to Y$ be a perfect onto map. If X is a σ-relatively discrete set, then so is Y.

One characterization of a *weakly* θ-*refinable* space is that each open cover has a σ-relatively discrete refinement [1], so the following result is also an immediate corollary of the theorem.

COROLLARY 2: [2]Let $f : X \to Y$ be a perfect onto map. If X is weakly θ-refinable, then so is Y.

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Tameness and Movability in Proper Shape Theory

ZVONKO ČERIN[a] AND GIAN MARIO GIANELLA[b]

[a]*Department of Mathematics*
University of Zagreb
Zagreb, 10001 Croatia
and
[b]*Dipartimento di Matematica*
Università di Torino
Torino, 11023 Italia

ABSTRACT: We study properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame and properly $M^{\mathcal{B}}$-movable spaces, where $\mathcal{B}$ and $\mathcal{C}$ denote classes of topological spaces. Both proper tameness and proper movability are invariants of a recently invented first author's proper shape theory and are described by the use of proper multivalued functions. The first is analogous to L. Siebenmann's notion of a tame at infinity space while the second is modelled on Borsuk's concept of movability in shape theory.

1. INTRODUCTION

The notions and results in this paper belong to the part of topology that could be described as proper shape theory. As shape theory is an improved homotopy theory designed to handle more successfully complicated spaces so is proper shape theory a modification of proper homotopy theory made with the same goal to provide us with a new insight into global properties even of those spaces for which the classical proper homotopy gives doubtful information.

In [5] the first author has described proper shape category of all topological spaces using Sanjurjo's method of multivalued functions from [11]. His approach was formally very similar to the one taken by Ball and Sher [2]. Instead of proper fundamental nets he considered proper multinets. The other steps were identical. He defined a notion of a proper homotopy for proper multinets and took for the morphisms of the proper shape category Sh_p proper homotopy classes of proper multinets.

The present paper is the third in a series begun by [5] where we shall study proper shape theory utilizing small proper multivalued functions. In the second [6] the first author considers certain morphisms of the category Sh_p called surjections, injections, and bijections, while here we shall introduce and investigate proper shape invariants related to shape dimension and movability from shape theory [4]. In other words, we shall transfer into proper shape theory from shape

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[c]E-mail: zcerin@x400.srce.hr.
[d]E-mail: gianella@dm.unito.it.

theory concepts of tameness and movability. The fourth part of the series [7] will do a similar thing with the first author's properties of smoothness and calmness.

Let us describe the content of the paper in greater detail. In Section 2 we recall notions and results from [5] that are necessary in further developments. The next Section 3 studies properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces. The origin for this notion could be traced back to L. Siebenmann's tame at infinity spaces [12]. The idea is that we require that small enough proper multivalued functions from members of a class of spaces $\mathcal{B}$ into a given space X factor through some member of another class C also through sufficiently small proper multivalued functions. This concept is related to the notion of shape dimension [4] and could be regarded as a substitute for it in the proper shape theory. We prove that this is an invariant in the category Sh_p, explore the role of classes $\mathcal{B}$ and C, and study what kind of maps will preserve and inversely preserve properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces. The classes of proper $M^{\mathcal{B}}$-surjections and proper $M^{\mathcal{B}}$-injections mentioned above are of key importance here.

In the following Section 4 a special case when C is the class $\mathcal{T}$ of all trees (which we call properly $M^{\mathcal{B}}$-contractible spaces) are related to Sher's results. Then we move on to consider in Section 5 properly $M^{\mathcal{B}}$-movable spaces which are the analogue of movable compacta [4] in proper shape theory.

Finally, in Section 6 we consider dependence of these notions on classes $\mathcal{B}$ and C under the assumption that they are connected with each other by morphisms from [6].

2. PRELIMINARIES ON PROPER SHAPE THEORY

Let X and Y be topological spaces. By a *multivalued function* $F: X \to Y$ we mean a rule which associates a nonempty subset $F(x)$ of Y to every point x of X. A multivalued function $F: X \to Y$ is *S-proper* provided for every compact subset C of Y its small counterimage $F'(C) = \{x \in X \mid F(x) \subset C\}$ is a compact subset of X. On the other hand, F is *B-proper* provided for every compact subset C of Y its big counterimage $F''(C) = \{x \in X \mid F(x) \cap C \neq \varnothing\}$ is a compact subset of X. We shall use the term *proper* to name either S-proper or B-proper. However, in a given situation, once we make a selection between two different kinds of properness it is understood that it will be retained throughout.

Observe that for single-valued functions the two notions of properness coincide. Classes of S-proper and B-proper multivalued functions are completely unrelated [5]. It follows that each of our notions and results involving proper multivalued functions actually has two versions.

Let $\mathrm{Cov}(Y)$ denote the collection of all numerable covers of a topological space Y (see [1]). With respect to the refinement relation $>$ the set $\mathrm{Cov}(Y)$ is a directed set. Two numerable covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a numerable cover and it's equivalence class by the same symbol. Consequently, $\mathrm{Cov}(Y)$ also stands for the associated quotient set.

Let $\sigma \in Y$. Let σ^+ denote the set of all numerable covers of Y refining σ while σ^* denotes the set of all numerable covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all numerable covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Let $\mathrm{Inc}(Y)$ denote the collection of all finite subsets c of $\mathrm{Cov}(Y)$ which have a unique (with respect to the refinement relation) maximal element $[c] \in \mathrm{Cov}(Y)$. We consider $\mathrm{Inc}(Y)$ ordered by the inclusion relation and regard $\mathrm{Cov}(Y)$ as a subset of single-element subsets of $\mathrm{Inc}(Y)$. Notice that $\mathrm{Inc}(Y)$ is a cofinite directed set.

For our approach to proper shape theory the following notion of size for multivalued functions will play the most important role.

Let $F: X \twoheadrightarrow Y$ be a multivalued function and let $\alpha \in \mathrm{Cov}(X)$ and $\gamma \in \mathrm{Cov}(Y)$. We shall say that F is an *(α, γ)-function* provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subset C_A$. On the other hand, F is *γ-small* provided there is an $\alpha \in \mathrm{Cov}(X)$ such that F is an (α, γ)-function. For a σ-small multivalued function $F: X \twoheadrightarrow Y$ we use $S(F, \sigma)$ to denote the family of all numerable covers α of X such that F is an (α, σ)-function.

Next we introduce the notions which correspond to the equivalence relation of proper homotopy for proper maps.

Let F and G be proper multivalued functions from a space X into a space Y and let γ be a numerable cover of Y. We shall say that F and G are *properly γ-homotopic* and write $F \overset{\gamma}{\simeq} G$ provided there is a γ-small proper multivalued function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) \subset H(x, 0)$ and $G(x) \subset H(x, 1)$ for every $x \in X$. We shall say that H is a *proper γ-homotopy* that *joins* F and G or that it *realizes* the relation (or proper homotopy) $F \overset{\gamma}{\simeq} G$.

The following lemma from [5] is crucial because it provides an adequate substitute for the transitivity of the relation of proper homotopy.

LEMMA 2.1: Let F, G, and H be proper multivalued functions from a space X into a space Y. Let $\sigma \in Y$ and $\tau \in \sigma^*$. If $F \overset{\tau}{\simeq} G$ and $F \overset{\tau}{\simeq} H$, then $F \overset{\sigma}{\simeq} H$.

The proof of Lemma 2.1 requires an interesting proposition [8, p. 358] on numerable covers of the product $X \times I$ of a space X with the unit segment I. We assume that the reader is familiar with this result and the notion of a stacked covering of $X \times I$ over a numerable cover of X. For a numerable cover σ of $X \times I$, we shall use $D(X, \sigma)$ to denote the collection of all numerable covers τ of X such that some stacked covering of $X \times I$ over τ refines σ. As a consequence of the above proposition, this collection is always nonempty.

The following two definitions correspond to Ball and Sher's definitions of proper fundamental net and proper homotopy for proper fundamental nets from [2].

Let X and Y be topological spaces. By a *proper multinet* from X into Y we shall mean a collection $\varphi = \{F_c | c \in \mathrm{Inc}(Y)\}$ of proper multivalued functions $F_c: X \twoheadrightarrow Y$ such that for every $\gamma \in \mathrm{Cov}(Y)$ there is a $c \in \mathrm{Inc}(Y)$ with $F_d \overset{\gamma}{\simeq} F_c$ for every $d > c$. We use functional notation $\varphi: X \twoheadrightarrow Y$ to indicate that φ is a proper multinet from X into Y. Let $M_p(X, Y)$ denote all proper multinets $\varphi: X \twoheadrightarrow Y$.

Two proper multinets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between topological spaces X and Y are *properly homotopic* and we write $\varphi \simeq \psi$ provided for every $\gamma \in \mathrm{Cov}(Y)$ there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\simeq} G_d$ for every $d > c$. On the other hand, we write $\varphi \overset{\gamma}{\simeq} \psi$

and call φ and ψ *properly* γ*-homotopic* provided there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\simeq} G_d$ for every $d > c$.

It follows from Lemma 2.1 that the relation of proper homotopy is an equivalence relation on the set $M_p(X, Y)$. The proper homotopy class of a proper multinet φ is denoted by $[\varphi]$ and the set of all proper homotopy classes by $Sh_p(X, Y)$.

Our goal now is to define a composition for proper homotopy classes of proper multinets. Let $\varphi = \{F_c\}: X \to Y$ be a proper multinet. Let φ: $\mathrm{Inc}(Y) \to \mathrm{Inc}(Y)$ be an increasing function such that for every $c \in \mathrm{Inc}(Y)$ the relation $d, e > \varphi(c)$ implies the relation $F_d \overset{[c]}{\simeq} F_e$. Here we make an assumption that an increasing function φ from a partially ordered set P into itself always satisfies the condition that $\varphi(p) > p$ for every $p \in P$. Let $C = \{(c, d, e) |\ c \in \mathrm{Inc}(Y),\ d,\ e > \varphi(c)\}$. Then C is a subset of $\mathrm{Inc}(Y) \times \mathrm{Inc}(Y) \times \mathrm{Inc}(Y)$ that becomes a cofinite directed set when we define that $(c, d, e) > (c', d', e')$ iff $c > c'$, $d > d'$, and $e > e'$. We shall use the same notation φ for an increasing function φ: $C \to \mathrm{Cov}(X \times I)$ such that F_d and F_e are joined by a proper $(\varphi(c, d, e), [c])$ homotopy whenever $(c, d, e) \in C$. Let $\overline{\varphi}$: $C \to X$ be an increasing function such that $[\overline{\varphi}(c, d, e)] \in D(X, \varphi(c, d, e))$ for every $(c, d, e) \in C$. In [5] it was proved that there is an increasing function φ^*: $\mathrm{Inc}(Y) \to \mathrm{Inc}(X)$ such that (1) $\varphi^*(c) > \overline{\varphi}(c, \varphi(c), \varphi(c))$ for every $c \in \mathrm{Inc}(Y)$, and (2) φ^* is cofinal in $\overline{\varphi}$, i.e., for every $(c, d, e) \in C$ there is an $m \in \mathrm{Inc}(Y)$ with $\varphi^*(m) > \overline{\varphi}(c, d, e)$. With the help of functions φ and φ^* we shall define the composition of proper homotopy classes of proper multinets as follows.

Let $\varphi = \{F_c\} : X \to Y$ and $\psi = \{G_s\} : Y \to Z$ be proper multinets. Let $\chi = \{H_s\}$, where $H_s = G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}$ for every $s \in \mathrm{Inc}(Z)$. Observe that each H_s is a proper multivalued function because the composition of two proper multivalued functions is a proper multivalued function. In [5] it was proved that the collection χ is a proper multinet from X into Z. We now define the composition of proper homotopy classes of proper multinets by the rule $[\{G_s\}] \circ [\{F_c\}] = [\{G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}\}]$. This composition of proper homotopy classes of proper multinets is well-defined and associative.

For a topological space X, let $\iota^X = \{I_a\}: X \to X$ be the identity proper multinet defined by $I_a = id_X$ for every $a \in \mathrm{Inc}(X)$. It is easy to show that for every proper multinet φ: $X \to Y$, the following relations hold:

$$[\varphi] \circ [\iota^X] = [\varphi] = [\iota^Y] \circ [\varphi].$$

We can summarize the above with the following main result from [5].

THEOREM 2.2: The topological spaces as objects together with the proper homotopy classes of proper multinets as morphisms and the composition of proper homotopy classes form the proper shape category Sh_p.

The above constructions may be done without any reference to S-proper and B-proper multivalued functions. In this way we shall get the shape category Sh. On the other hand, in both cases, we may require that all multivalued functions belong to a class of multivalued functions which is closed with respect to pastings from the proof of Lemma 2.1 in [5] and compositions. In particular, we may assume that they are either upper semicontinuous or lower semicontinuous.

3. PROPERLY $M_{\mathcal{C}}^{\mathcal{B}}$-TAME SPACES

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. A space X is *properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ such that for every $B \in \mathcal{B}$ and every proper τ-small multivalued function $F: B \to X$ there is a $C \in \mathcal{C}$ and a proper σ-small multivalued function $H: C \to X$ with the property that for every $\alpha \in \mathrm{Cov}(C)$ there is a proper α-small multivalued function $G: B \to C$ with $F \overset{\sigma}{\simeq} H \circ G$. A class of spaces is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame provided each member of it is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

We shall first show that the property of being properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame is a proper shape invariant, i.e., that if X and Y are equivalent objects of the category Sh_p and X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame then Y is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame. In fact, a much better result is true. The properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces are preserved under the following weak form of domination.

A class of spaces $\mathcal{B}$ is *properly M-dominated* by a class of spaces $\mathcal{A}$ provided for every $B \in \mathcal{B}$ and every $\beta \in \mathrm{Cov}(B)$ there is an $A \in \mathcal{A}$ and a proper β-small multivalued function $G: A \to B$ such that for every $\alpha \in \mathrm{Cov}(A)$ we can find a proper α-small multivalued function $F: B \to A$ with $G \circ F \overset{\beta}{\simeq} id_B$.

THEOREM 3.1: A space X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame if and only if it is properly M-dominated by a class of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces.

Proof: Since every space properly M-dominates itself, it remains to prove the "if" part. Let a numerable cover σ of X be given. Let $\eta \in \sigma^*$. By assumption, there is a properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame space Y and a proper η-small multivalued function $G: Y \to X$ such that for every $\varepsilon \in \mathrm{Cov}(Y)$ there is a proper ε-small multivalued function $F: X \to Y$ with

$$G \circ F \overset{\eta}{\simeq} id_X. \tag{1}$$

Let $\delta \in S(G, \eta)$. Since Y is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame, there is an $\varepsilon \in \mathrm{Cov}(Y)$ such that for every $B \in \mathcal{B}$ and every proper ε-small multivalued function $D: B \to Y$ there is a $C \in \mathcal{C}$ and a proper δ-small multivalued function $L: C \to Y$ so that for every $\pi \in \mathrm{Cov}(C)$ there is a proper π-small multivalued function $H: B \to C$ with

$$D \overset{\delta}{\simeq} L \circ H. \tag{2}$$

Pick a proper ε-small multivalued function F such that (1) holds. Let M be a proper η-homotopy that realizes the relation (1). Let $\zeta \in D(M, \eta)$ and $\theta \in S(F, \varepsilon)$. Let $\tau \in \mathrm{Cov}(X)$ be a common refinement of ζ and θ. Then τ is the required numerable cover.

To verify this, consider a member B of $\mathcal{B}$ and a proper τ-small multivalued function $K: B \to X$. The composition D of K and F is a proper ε-small multivalued function from B into Y. Pick C and L as above and let N denote the composition $G \circ L$. Observe that N is a proper σ-small multivalued function.

Let π be a numerable cover of C. Choose a proper π-small multivalued function H so that (2) is true. Let P be a proper δ-homotopy joining D and $L \circ H$. Then

$$K \overset{\eta}{\simeq} G \circ F \circ K. \tag{3}$$

because $M \circ (K \times id_I)$ is a proper η-homotopy joining K and $G \circ F \circ K$, and

$$G \circ F \circ K \overset{\eta}{\simeq} N \circ H. \tag{4}$$

because $G \circ P$ is a proper η-homotopy joining $G \circ F \circ K$ and $N \circ H$. The relations (3) and (4) together imply $K \overset{\sigma}{\simeq} N \circ H$. ❑

The proper M-domination is weaker than the quasi-Sh_p-domination and thus also weaker than Sh_p-domination. Recall that a class of spaces $\mathcal{A}$ is *Sh_p-dominated* or *properly shape dominated* by a class of spaces $\mathcal{B}$ provided for every $X \in \mathcal{A}$ there is a $Y \in \mathcal{B}$ and proper multinets $\varphi: X \to Y$ and $\psi: X \to Y$ with the composition $\psi \circ \varphi$ properly homotopic to the identity proper multinet ι^X on X. On the other hand, $\mathcal{A}$ is *quasi-Sh_p-dominated* by $\mathcal{B}$ provided for every $X \in \mathcal{A}$ and every $\sigma \in \mathrm{Cov}(X)$ there is a $Y \in \mathcal{B}$ and proper multinets $\varphi: X \to Y$ and $\psi: Y \to X$ with the composition $\psi \circ \varphi$ properly σ-homotopic to the identity proper multinet ι^X on X. The notion of quasi-Sh_p-domination is analogous to notion of quasidomination in [3].

It was proved in [6] that the quasi-Sh_p-domination is stronger than the proper M-domination. Observe that the quasi-Sh_p-domination is clearly weaker than the Sh_p-domination. Therefore, we obtain the following consequence.

COROLLARY 3.2: A space is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame iff it is either Sh_p-dominated or quasi-Sh_p-dominated by a class of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces.

Another example of proper M-domination provides the notion of being properly $\mathcal{B}$-like. Recall that a space X is *properly $\mathcal{B}$-like*, where $\mathcal{B}$ is a class of spaces, provided for every $\sigma \in \mathrm{Cov}(X)$ there is a member Y of $\mathcal{B}$ and a proper map f: $X \to Y$ such that the inverse $f^{-1}: X \to Y$ is a proper σ-small multivalued function. In [6] we showed that if a space X is properly $\mathcal{B}$-like, then X is properly M-dominated by $\mathcal{B}$. Hence, we get the following conclusion.

COROLLARY 3.3: A space X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame iff it is properly $\mathcal{D}$-like, where $\mathcal{D}$ is a class of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces.

In the next two theorems we explore in which way does the definition of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces depend on classes $\mathcal{B}$ and $\mathcal{C}$. Their proofs are left to the reader.

THEOREM 3.4: Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If a space X is both properly $M_{\mathcal{A}}^{\mathcal{B}}$-tame and properly $M_{\mathcal{C}}^{\mathcal{A}}$-tame, then X is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

THEOREM 3.5: Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are properly M-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is properly $M_{\mathcal{D}}^{\mathcal{A}}$-tame, then it is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

COROLLARY 3.6: Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are (quasi) Sh_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is properly $M_{\mathcal{D}}^{\mathcal{A}}$-tame, then it is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

The following weak form of the notion of being properly $\mathcal{B}$-like is more in line with our point of view because it is based on multivalued functions. It offers us the possibility to improve Corollary 3.6 in Theorem 3.7 with a similar proof.

Let C be a class of topological spaces. A space X is *properly* M^C*-like* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a member Y of C and a numerable cover α of Y such that for every $\beta \in \mathrm{Cov}(Y)$ there is a proper β-small multivalued function $F: X \to Y$ such that F^{-1} is a proper (α, σ)-function.

THEOREM 3.7: A space X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame iff it is properly $M^{\mathcal{D}}$-like, where $\mathcal{D}$ is a class of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces.

In the rest of this section we shall address the question of identifying those proper maps which will preserve or inversely preserve properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces. The answer provide proper maps studied in [6] whose definitions we now recall.

Let $\mathcal{B}$ be a class of topological spaces. A proper map $f: X \to Y$ is a *proper* $M^{\mathcal{B}}$*-surjection* provided for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is a $\rho \in \mathrm{Cov}(Y)$ such that for every proper ρ-small multivalued function F from a member of $\mathcal{B}$ into Y there is a proper σ-small multivalued function G with $F \overset{\tau}{\simeq} f \circ G$.

A special case of proper $M^{\mathcal{B}}$-surjections are *properly right M-placid* maps, i.e., proper maps $f: X \to Y$ such that for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is a proper σ-small multivalued function $J: Y \to X$ with $f \circ J \overset{\tau}{\simeq} id_Y$. In fact, every properly right M-placid proper map is a proper M^S-surjection, where S denotes the class of all topological spaces.

Observe that a proper map $f: X \to Y$ which has a right proper homotopy inverse (i.e., for which there is a proper map $g: Y \to X$ with $f \circ g$ properly homotopic to id_Y) is properly right M-placid. The same is true if the proper map has a right Sh_p-inverse.

Another important example of properly right M-placid maps provide properly refinable maps. We call an onto proper map $f: X \to Y$ between topological spaces *properly refinable* provided for every numerable cover τ of Y and every numerable cover σ of X there is an onto proper map $g: X \to Y$ such that f and g are τ-close and g^{-1} is a proper σ-small multivalued function. We call g a proper (σ, τ)*-refinement* of the map f. The notion of a refinable map between compact metric spaces was first defined by Jo Ford and James Rogers, Jr. The above extension to arbitrary topological spaces is particularly suitable for our theory.

LEMMA 3.8: Properly refinable maps are properly right M-placid.

Proof: Let $f: X \to Y$ be a properly refinable map. Let σ and τ be numerable covers of X and Y, respectively. Let $\pi \in \tau^*$. Let a numerable cover ν of X be a common refinement of $f^{-1}(\pi)$ and σ. Let $g: X \to Y$ be a proper (ν, π)-refinement of f. Observe that g^{-1} is a proper σ-small multivalued function, the composition $f \circ g^{-1}$ is π-small, and since f and g are π-close for every $y \in Y$, there is a member of π which contains y and intersects $f \circ g^{-1}(y)$. It follows that the function $H: Y \times I \to Y$ defined by $H(y, t) = \{y\} \cup f \circ g^{-1}(y)$ for every $(y, t) \in Y \times I$ is a proper τ-homotopy joining id_Y and $f \circ g^{-1}$. ❑

The following result shows that properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces are preserved under proper $M^{\mathcal{B}}$-surjections.

THEOREM 3.9: If $f: X \to Y$ is a proper $M^{\mathcal{B}}$-surjection and the domain X is a properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame space, then the codomain Y is also a properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame space.

Proof: Let a numerable cover σ of Y be given. Let $\mu \in \sigma^*$ and put $\alpha = f^{-1}(\mu)$. Since X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame, there is a $\beta \in \mathrm{Cov}(X)$ such that for every $B \in \mathcal{B}$ and every prope β-small multivalued function $K: B \to X$ there is a $C \in C$ and a proper α-small multivalued function $L: C \to X$ so that for every $\gamma \in \mathrm{Cov}(C)$ we can find a proper γ-small multivalued function $G: B \to C$ with

$$K \overset{\alpha}{\simeq} L \circ G. \tag{1}$$

Finally, we utilize the fact that f is a proper $M^{\mathcal{B}}$-surjection to select the required numerable cover τ of Y so that for every proper τ-small multivalued function F from a member B of $\mathcal{B}$ into Y there is a proper β-small multivalued function K: $B \to X$ with

$$F \overset{\mu}{\simeq} f \circ K. \tag{2}$$

Consider a $B \in \mathcal{B}$ and a proper τ-small multivalued function $F: B \to Y$. Choose a K and then C and L as above. Let H denote the composition of L and f. Notice that H is a proper σ-small multivalued function. At last, for a given $\gamma \in \mathrm{Cov}(C)$, pick a G as above. Then from (1) and (2) we obtain the following chain of relations

$$F \overset{\mu}{\simeq} f \circ K \overset{\mu}{\simeq} f \circ L \circ G = H \circ G.$$

Hence, $F \overset{\sigma}{\simeq} H \circ G$ and Y is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame. ❑

The existence of a properly refinable map from a space X onto a space Y clearly implies that X is properly $M^{\{Y\}}$-like. Hence, as a consequence of the above Lemma 3.8 and Theorems 3.7 and 3.9 we obtain the following proper version of parts (3) and (5) of Theorem 1.8 in Kato [9].

COROLLARY 3.10: Let $f: X \to Y$ be a properly refinable map. Then X is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame if and only if Y is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

For the inverse preservation of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces from the codomain to the domain we must assume that the map f is either properly left M-placid or that it is a proper $M_{\mathcal{C}}^{\mathcal{B}}$-bijection. Let us recall the definitions of these notions from [6].

A proper map $f: X \to Y$ is *properly left M-placid* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a proper σ-small multivalued function $J: Y \to X$ such that $J \circ f \overset{\sigma}{\simeq} id_X$.

Let $\mathcal{B}$ be a class of topological spaces. A proper map $f: X \to Y$ is called a *proper M^{C}-injection* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ and a $\xi \in \mathrm{Cov}(Y)$ such that for proper τ-small multivalued functions F and G from a member B of $\mathcal{B}$ into X the relation $f \circ F \overset{\xi}{\simeq} f \circ G$ implies the relation $F \overset{\sigma}{\simeq} G$.

At last, for classes $\mathcal{B}$ and C of spaces, a proper map which is both a proper $M^{\mathcal{B}}$-injection and proper M^{C}-surjection will be called a *proper* $M_{\mathcal{C}}^{\mathcal{B}}$-bijection. We shall use a shorter name *proper $M^{\mathcal{B}}$-bijection* for a proper $M_{\mathcal{B}}^{\mathcal{B}}$-bijection.

Observe that every proper map $f: X \to Y$ which has a left proper homotopy inverse (i.e., for which there is a proper map $g: Y \to X$ with the composition $g \circ f$ properly homotopic to id_X) is properly left M-placid. The same is true if the map

f has a left Sh_p-inverse. Moreover, a properly left M-placid proper map is a proper M^S-injection, where S denotes the class of all topological spaces.

The following two theorems could be proved with the above techniques.

THEOREM 3.11: If a map $f: X \to Y$ is properly left M-placid and the codomain Y is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame, then the domain X is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

An important example of properly left M-placid maps provide inclusions $i_{A,X}$ of the proper M-retracts A of a space X. Here, we will say that a closed subset A of a space X is a *proper M-retract* of X provided for every numerable cover σ of A there is a proper σ-small multivalued function $R: X \to A$ such that $a \in R(a)$ for every $a \in A$. Hence, the following is a consequence of the Theorem 3.11.

COROLLARY 3.12: A proper M-retract of a properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame space is itself properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

THEOREM 3.13: If a map $f: X \to Y$ is a proper $M_{\mathcal{C}}^{\mathcal{B}}$-bijection and the codomain Y is properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame, then the domain X is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

4. PROPERLY $M^{\mathcal{B}}$-CONTRACTIBLE SPACES

An important special case of properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame spaces is when the class C is the class $\mathcal{T}$ of all trees. Here, by a *tree* we mean a locally finite, connected, and simply connected simplicial 1-complex.

Let $\mathcal{B}$ be a class of spaces. A space X is *properly $M^{\mathcal{B}}$-contractible* provided it is properly $M_{\mathcal{T}}^{\mathcal{B}}$-tame.

Since every tree is a locally compact ANR, it is easy to see that a space X is properly $M^{\mathcal{B}}$-contractible iff for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ such that for every proper τ-small multivalued function F from a member Z of $\mathcal{B}$ into X there is a tree T, a proper map $g: Z \to T$, and a proper σ-small multivalued function $H: T \to X$ with $F \overset{\sigma}{\simeq} H \circ g$.

An important special case is the following. A space X is *properly M-contractible* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a tree T, a proper map $g: X \to T$, and a proper σ-small multivalued function $H: T \to X$ with $id_X \overset{\sigma}{\simeq} H \circ g$.

For the next theorem we must also recall definitions of internally properly calm and internally properly movable spaces. They illustrate that most concepts in proper shape theory have appropriate internal versions.

A space X is *properly internally calm* provided there is a $\sigma \in \mathrm{Cov}(X)$ such that proper maps into X which are properly σ-homotopic are properly homotopic.

A space X is *properly internally movable* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ such that every τ-small proper multivalued function into X is properly σ-homotopic to a proper map.

THEOREM 4.1: Let X be a space which is both internally properly calm and internally properly movable. Then X is properly M-contractible if and only if X is properly homotopically dominated by a tree.

Proof: The "if" part is easy and it is true without any assumptions about the space X. In order to prove the "only if" part, choose numerable covers σ, τ, and

ρ of X such that every two proper maps into X which are properly σ-homotopic are properly homotopic, $\tau \in \sigma^*$, $\rho \in \tau^+$, and every proper ρ-small multivalued function into X is properly τ-homotopic to a proper map. Since X is properly M-contractible, there is a tree T, a proper map $g: X \to T$, and a proper ρ-small multivalued function $H: T \to X$ with $id_X \overset{\rho}{\simeq} H \circ g$. By assumption, there is a proper map $h: T \to X$ such that $h \overset{\tau}{\simeq} H$. It follows that $h \circ g \overset{\tau}{\simeq} H \circ g$ and $id_X \overset{\sigma}{\simeq} h \circ g$. Hence, id_X and $h \circ g$ are properly homotopic and X is properly homotopically dominated by the tree T. ❑

COROLLARY 4.2: A locally compact ANR is properly M-contractible if and only if it has property SUV^∞.

Proof: This follows from Theorem 4.1 and the property (*) in [10, p.243]. ❑

5. PROPERLY $M^{\mathcal{B}}$-MOVABLE SPACES

In the present section we shall transfer from shape theory into proper shape theory the important invariant of movability. This concept was invented by Borsuk [4] for compact metric spaces. We shall define properly $M^{\mathcal{B}}$-movable spaces with respect to a class $\mathcal{B}$ of spaces in order to cover all possible variations of movability (see [4]). In this and the next section all proofs are omitted since they resemble proofs in previous sections.

Let $\mathcal{B}$ be a class of topological spaces. A space X is *properly $M^{\mathcal{B}}$-movable* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ such that for every $B \in \mathcal{B}$, every proper τ-small multivalued function $F: B \to X$, and every $\rho \in \mathrm{Cov}(X)$ there is a proper ρ-small multivalued function $G: B \to X$ with $F \overset{\sigma}{\simeq} G$.

We shall first consider how this definition depends on the class $\mathcal{B}$. Once again the proper M-domination comes into play.

THEOREM 5.1: If a class of spaces $\mathcal{C}$ properly M-dominates another such class $\mathcal{B}$ and a space X is properly $M^{\mathcal{C}}$-movable, then X is also properly $M^{\mathcal{B}}$-movable.

Our goal now is to show that proper $M^{\mathcal{C}}$-movability is indeed a proper shape invariant. We can prove a far better result, namely that it is preserved under quasi-Sh_p-domination.

THEOREM 5.2: A space X is properly $M^{\mathcal{C}}$-movable if and only if it is quasi-Sh_p-dominated by a class of properly $M^{\mathcal{C}}$-movable spaces.

The following result is typical for shape theory. It illustrates the role of properly $M^{\mathcal{B}}_{\mathcal{C}}$-tame spaces and is similar to Borsuk's theorem that an n-movable compactum of shape dimension at most n is movable [4].

THEOREM 5.3: Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. If a space X is at the same time properly $M^{\mathcal{B}}_{\mathcal{C}}$-tame and properly $M^{\mathcal{C}}$-movable, then it is also properly $M^{\mathcal{B}}$-movable.

The following special case of Theorem 5.3 is worth mentioning. In order to state it, we shall first introduce the notion of proper M^{C_p}-extensors.

Let C_p be a class of pairs (A,B) of topological spaces with B a subspace of A. Let classes $\{A \mid (A,B) \in C_p \text{ for some } B\}$ and $\{B \mid (A,B) \in C_p \text{ for some } A\}$ be denoted by C_p' and C_p''.

A space X is a proper *M^{C_p}-extensor* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ with the property that for every member (A,B) of C_p and every proper τ-small multivalued function $F\colon B \to X$ there is a proper σ-small multivalued function $G\colon A \to X$ with $F = G|_B$.

One can easily check that a proper M^{C_p}-extensor is properly $M^{C_p''}_{C_p'}$-tame. Hence, we obtain the following corollary.

COROLLARY 5.4: Let Cp be a class of proper pairs. If a space X is both properly $M^{C_p'}$-movable and a proper M^{C_p}-extensor, then the space X is also properly $M^{C_p''}$-movable.

In the rest of this section we shall consider the question of identifying those proper maps which will preserve or inversely preserve properly $M^{\mathcal{B}}$-movable spaces. The answer provides proper maps studied in [6] whose definitions have been recalled in Section 3. The following result resembles Theorem 3.9.

THEOREM 5.5: If a map $f\colon X \to Y$ is a proper M^{C}-surjection and the domain X is properly M^{C}-movable, then the codomain Y is also properly M^{C}-movable.

COROLLARY 5.6: The image under a properly refinable map of a properly M^{C}-movable space is properly M^{C}-movable.

In an attempt to prove an analogue of Theorem 3.11 for properly M^{C}-movable spaces instead of properly left M-placid we must use the following stronger form of this notion.

A proper map $f\colon X \to Y$ between topological spaces is *properly left M_s-placid* provided for every numerable cover σ of X there is a numerable cover α of Y and a numerable cover τ of X such that for every numerable cover ρ of X there is a ρ-small proper (α, σ)-function $J\colon Y \to X$ and a proper σ-homotopy H joining $J \circ f$ and id_X with $\tau \in D(H, \sigma)$.

THEOREM 5.7: If a proper map $f\colon X \to Y$ is properly left M_s-placid and Y is properly M^{C}-movable, then X is also properly M^{C}-movable.

COROLLARY 5.8: A proper M-retract of a properly M^{C}-movable space is itself properly M^{C}-movable.

THEOREM 5.9: If $f\colon X \to Y$ is a proper M^{C}-bijection and the codomain Y is properly M^{C}-movable, then the domain X is also properly M^{C}-movable.

6. COVERED AND EXTENDED CLASSES

In this section we shall explore dependence of all proper shape invariants which were defined on classes of spaces involved under the assumption that these classes are connected by either surjections or injections. The connection can be through one of the following two notions.

Let $\mathcal{F}$ be a class of proper maps and let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. We shall say that the class $\mathcal{C}$ is *$\mathcal{F}$-covered* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and an $h: B \to C$ from $\mathcal{F}$. Similarly, the class $\mathcal{C}$ is *$\mathcal{F}$-extended* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and a $k: C \to B$ from $\mathcal{F}$.

For a class of spaces $\mathcal{B}$ we shall use $\mathcal{B}_i$, $\mathcal{B}_s$, and $\mathcal{B}_b$ to denote the classes of all proper $M^{\mathcal{B}}$-injections, proper $M^{\mathcal{B}}$-surjections, and proper $M^{\mathcal{B}}$-bijections. Also, $\mathcal{B}^i$, $\mathcal{B}^s$, and $\mathcal{B}^b$ denote the classes of all proper $N^{\mathcal{B}}$-injections, proper $N^{\mathcal{B}}$-surjections, and proper $N^{\mathcal{B}}$-bijections. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are classes of maps we let $\mathcal{F}\mathcal{G}$ denote the intersection $\mathcal{F} \cap \mathcal{G}$.

THEOREM 6.1: Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces. If a space X properly $M_{\mathcal{D}}^{\mathcal{A}}$-tame and either

(cc) $\mathcal{B}$ is $\mathcal{C}^s\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\{X\}^i$-covered by $\mathcal{D}$,

(ce) $\mathcal{B}$ is $\mathcal{C}^s\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^s$-extended by $\mathcal{D}$,

(ec) $\mathcal{B}$ is $\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\{X\}^s$-covered by $\mathcal{D}$, or

(ee) $\mathcal{B}$ is $\mathcal{C}^s\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^s$-extended by $\mathcal{D}$,

then X is also properly $M_{\mathcal{C}}^{\mathcal{B}}$-tame.

THEOREM 8.2: Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. If a space X is properly $M^{\mathcal{B}}$-movable and the class $\mathcal{C}$ is either $\{X\}^s$-covered or $\{X\}^s$-extended by $\mathcal{B}$, then X is also properly $M^{\mathcal{C}}$-movable.

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The Jordan Curve-type Theorems for the Funnel in 2-dimensional Semiflows

KRZYSZTOF CIESIELSKI[a]

Mathematics Institute
Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

ABSTRACT: We investigate "the past" of a nonstationary point x in a semiflow $\pi: \mathbb{R}_+ \times M \to M$ on a 2-manifold M. In a natural way there are defined two boundary trajectories T_a and T_b; by the funnel we mean $F(x) = \{y: \pi(t, y) = x$ for some $t\}$ and we put $D(x) = F(x)\backslash(T_a \cup T_b)$. We show that if x is not a point of negative unicity, then $D(x)$ is homeomorphic to $\mathbb{R}^2$. Also, if T is a nonboundary trajectory through x, then $D(x)\backslash T$ has two components for a regular point x and $D(x)\backslash T$ has two or three components for a periodic point x.

1. INTRODUCTION

The behavior of solutions and topological properties of trajectories play a major role in the theory of topological dynamical systems (flows) and qualitative theory of differential equations. In this paper we give a very precise description of the topological properties of negative trajectories through a nonstationary point x in a semiflow on a 2-manifold.

In [5], the funnel cuts and sections of a point x, i.e. "the past" of a point x in time t (denoted by $F(t, x)$, which appears to be a point or an arc) and in interval time $[s, t]$, are characterized. This paper may be regarded as a continuation of investigations in [5]. Other applications of [5], paying attention to limit sets in semiflows on 2-manifolds, were shown in [4].

In this paper we investigate the "ways" along which we can reach a point x. On account of [5], we can single out two *boundary trajectories* which are naturally defined according to the properties of $F(t, x)$; roughly speaking, they are given by the end-points of the arcs $F(t, x)$ for $t \geq 0$. We investigate the set $D(x)$ which is "the past" of x without just these two boundary trajectories. We show that this set is homeomorphic to the plane.

Also, we prove that for the set $D(x)$ the Jordan curve-type theorems hold. Any negative trajectory through x cuts $D(x)$. If a point x is regular, then any nonboundary trajectory cuts $D(x)$ into two regions homeomorphic to the plane. If a point x is periodic, then any nonboundary trajectory cuts $D(x)$ into two or three regions, each of them homeomorphic to the plane. Such a trajectory $T(x)$ cuts $D(x)$ into three regions if and only if the (unique) positive trajectory through x is contained in $T(x)$ and it is equal neither to $T(x)$ nor to any boundary trajectory.

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[a]Supported by a Polish Scientific Grant 658/2/91. E-mail: ciesiels@im.uj.edu.pl.

In the particular situation where the phase space is equal to $\mathbb{R}^2$, these theorems were presented by the author in the University of Warwick preprint. The results presented in this paper are more general, but also the proofs given here are much simpler than those in the preprint.

Note that the theorems also hold for noncompact and nonorientable 2-manifolds.

2. PRELIMINARIES

By an *arc* (a *Jordan curve*) we mean a homeomorphic image of the compact interval $[-1, 1]$ (a unit circle). By $\overline{ab}$ we denote an arc with end-points a, b. For a given set X we denote its interior by $\mathrm{Int}X$, its closure by $\mathrm{Cl}X$ and its boundary by ∂X.

When we consider a function f defined on a set containing an interval (a, b), we denote the set $f((a, b))$ by $f(a, b)$. In the same way we introduce the symbols $f[a, b)$, $f(a, b]$ and $f[a, b]$.

A *semiflow* (*semidynamical system*) on M (which is called a *phase space*) is a triplet $(M, \mathbb{R}_+, \pi)$ where $\pi\colon \mathbb{R}_+ \times M \to M$ is a continuous function such that $\pi(0, x) = x$ and $\pi(t, \pi(u, x)) = \pi(t + u, x)$ for any t, u, x.

Assume that a semiflow $(M, \mathbb{R}_+, \pi)$ on a 2-manifold M (without boundary) is given.

By $\pi^+(x)$ we denote $\{\pi(t, x)\colon t \geq 0\}$ and call it the *positive trajectory through* x. We put $F(t, x) = \{y \in M\colon \pi(t, y) = x\}$, $F([u, v], x) = \bigcup\{F(t, x)\colon t \in [u, v]\}$ for $u < v$ and $F(x) = \bigcup\{F(t, x)\colon t \geq 0\}$; the last set is called the *funnel through* x.

A function $\sigma\colon (\alpha, 0] \to M$ is called a *negative solution through* x if $\sigma(0) = x$, $\pi(t, \sigma(u)) = \sigma(t + u)$ whenever $u, t + u \in (\alpha, 0]$, $t \geq 0$, and σ is maximal relative to the above properties, with respect to inclusion. Usually such functions are called maximal negative solutions; in this paper "negative solution" or "solution" means maximal negative solution. It is known [2] that every solution is continuous. For a solution $\sigma\colon (\alpha, 0] \to M$ we call its image $\sigma(\alpha, 0]$ a *negative trajectory through* x. According to the main results of [11] and [2, Theorem 11.8] we may assume without loss of generality that the domain of any negative solution is equal to $(-\infty, 0]$. This is because we can transform the system by a suitable isomorphism which does not change trajectories (and therefore their topological properties), but only changes the speed of movement along trajectories; note that such a reparametrization can change some of the dynamical properties of the system.

We put $L_\sigma^-(x) = \{y \in M\colon \sigma(t_n) \to y \text{ for some } t_n \to -\infty\}$, where σ is a negative solution through x and call it a *negative limit set*. Note that for a given point x, different negative solutions can give different negative limit sets.

A point x is said to be:

(a) *stationary* if $\pi(t, x) = x$ for every $t \geq 0$,

(b) *periodic* if there exists a $t > 0$ such that $\pi(t, x) = x$ and x is not stationary; the smallest t with the above property is called the *period* of x,

(c) *regular* if it is neither stationary nor periodic,

(d) *a point of negative unicity* if for any $t \geq 0$ the set $F(t, x)$ has precisely one element,

(e) *singular* if there exist $y_1 \neq y_2 \in \mathbb{R}^2$ and a $t > 0$ such that $\pi(t, y_1) = \pi(t, y_2) = x$ but $\pi(u, y_1) \neq \pi(u, y_2)$ for any $u \in [0, t)$.

Note that if x is a point of negative unicity then there is only one negative solution through x and if x is not a point of negative unicity then there exists a $\lambda \leq 0$ such that for any solutions σ_1, σ_2 through x and $t \in [\lambda, 0]$ we have $\sigma_1(t) = \sigma_2(t)$, moreover $\sigma_1(\lambda) = \sigma_2(\lambda)$ is a singular point.

For the basic properties of flows, semiflows and topological background used here the reader is referred to the books [2], [9], [13], [15].

Throughout this paper we assume that $(M, \mathbb{R}_+, \pi)$ is a given semiflow on a 2-manifold (without boundary) M. We admit also noncompact manifolds. We assume that $x \in M$ is a nonstationary point which is not a point of negative unicity (the funnel through a stationary point cannot be so well characterized—compare the examples in [5]; for a point of negative unicity the analogous description is trivial, compare Remark 3.4). Recall that we assume that any solution is defined on $(-\infty, 0]$, as this involves no loss of generality.

3. PREPARATORY THEOREMS

We start from a proposition which is an immediate corollary from [5, Theorem 3.4].

PROPOSITION 3.1: For a given point x there exists an $s \geq 0$ such that $F(t, x)$ is a point for $t \leq s$ and an arc for $t > s$.

DEFINITION 3.2: If x is a periodic point we denote by τ its period; if x is a regular point we denote by τ an arbitrarily fixed positive number. By λ we denote $\sup\{s\colon F(s, x) \text{ is a point}\}$.

Assume that a_1 and b_1 are the end-points of the arc $F(\lambda + \frac{\tau}{2}, x)$. If a_n and b_n are the end-points of the arc $F(\lambda + n\frac{\tau}{2}, x) = \overline{a_n b_n}$, then by a_{n+1} and b_{n+1} we denote the end-points of the arc $F(\lambda + (n+1)\frac{\tau}{2}, x)$ in such way that $\pi(\frac{\tau}{2}, a_{n+1}) = a_n$ and $\pi(\frac{\tau}{2}, b_{n+1}) = b_n$. We define $\sigma_a : (-\infty, 0] \to M$ by:

$$\sigma_a(-t) = \begin{cases} \pi(\lambda + \frac{\tau}{2} - t, a_1), & \text{for } t \in [0, \lambda + \frac{\tau}{2}], \\ \pi(\lambda + (n+1)\frac{\tau}{2} - t, a_{n+1}), & \text{for } t \in [\lambda + n\frac{\tau}{2}, \lambda + (n+1)\frac{\tau}{2}]. \end{cases}$$

The mapping σ_b is defined in an analogous way. It is very easy to verify that σ_a and σ_b are negative solutions. We call them *boundary solutions*, the ranges of these solutions will be called the *boundary trajectories*.

Roughly speaking, boundary trajectories are given by the end-points of the arcs $F(t, x)$.

NOTATION 3.3: We will investigate the topological properties of trajectories contained in the funnel through x. We denote:

(a) σ_1^x, σ_2^x — the boundary solutions through x;
(b) $T_1(x)$, $T_2(x)$ — the boundary trajectories given by the solutions σ_1^x and σ_2^x;
(c) $D(x) = F(x)\backslash(T_1(x) \cup T_2(x))$;
(d) $T_i([u, v], x) = \sigma_i^x[-v, -u]$ for $i = 1, 2$, $u < v$ (these sets are the segments of trajectories $T_1(x)$ and $T_2(x)$);
(e) if x is a periodic point, we denote by τ its period.

REMARK 3.4: If x was a point of negative unicity, then we could define boundary trajectories in an obvious way and we would get $T_1(x) = T_2(x) = F(x)$, so $D(x) = \varnothing$.

REMARK 3.5: For a periodic point we may distinguish three types of negative trajectories:
(a) homeomorphic to a circle (there is only the one solution which gives such negative trajectory, this trajectory is equal to $\pi^+(x)$),
(b) given by a solution σ, where σ is an injection,
(c) given by a solution σ, where σ is not an injection, but $\sigma(-\infty, 0]$ is not homeomorphic to a circle; the negative trajectory given by σ is a figure-of-six (compare [15]) and there is an $n \geq 1$ such that $\sigma|_{(-\infty,-n\tau]}$ is an injection and $\sigma[-n\tau, 0]$ is a circle.

The boundary trajectories may be only of types (a) or (b).

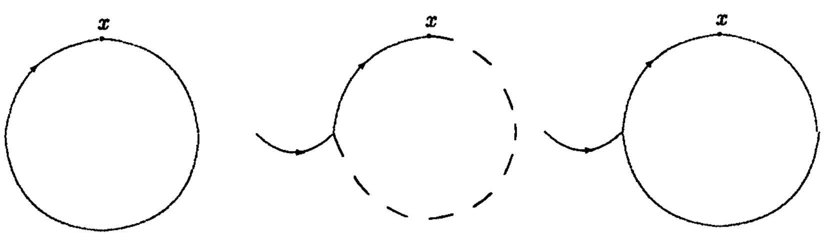

FIGURE 1

From Theorem 4.2 in [5] and the definition of boundary trajectories we immediately obtain the following two propositions:

PROPOSITION 3.6: Assume that x is a singular point; moreover, if x is periodic, we assume that $t < \tau$ (compare 3.3). Then $F([0, t], x)$ is homeomorphic to a triangle with the sides equal to $T_1([0, t], x)$, $T_2([0, t], x)$ and the arc $F(t, x)$.

PROPOSITION 3.7: Assume that x is a singular periodic point and $t > \tau$. Denote by a_t and b_t the end-points of the arc $F(t, x)$. Then $F([0, t], x)$ is homeomorphic to an annulus or a Möbius strip. Moreover, the boundary of this set is equal to $\pi([0, \tau], a_t) \cup \pi([0, \tau], b_t) \cup L$, where L is the largest subset of the arc $F(t, x)$ with $L \cap F([0, t), x) = \varnothing$.

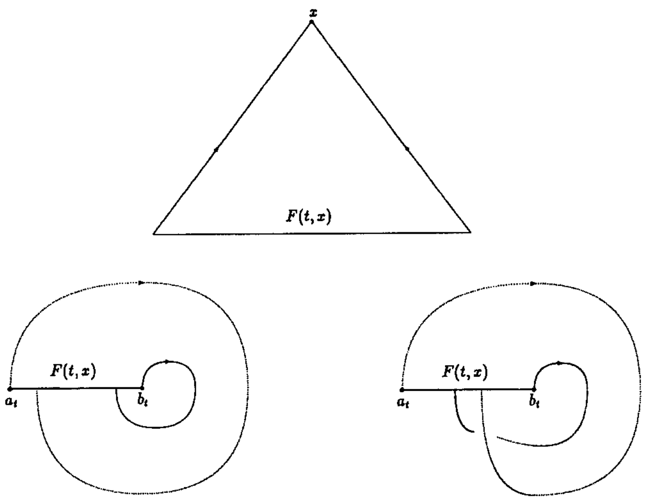

FIGURE 2

From 3.7 we get:

COROLLARY 3.8: If under the assumptions of Proposition 3.7 one of the boundary trajectories, say $T_1(x)$, is equal to $\pi^+(x)$ (which is homeomorphic to a circle), then $F([0, t], x)$ is homeomorphic to an annulus with one boundary circle equal to $\pi^+(x)$ and the second boundary circle equal to $T_2([t-\tau, t], x) \cup L$, where L is the largest subset of the arc $F(t, x)$ with $L \cap F([0, t), x) = \varnothing$.

Proof: Let a_t be the end-point of an arc $F(t, x)$ with $a_t \in \pi^+(x)$. According to 3.7, $\pi([0, \tau], a_t)$ is contained in $\partial F([0, t], x)$. The set $\pi([0, \tau], a_t) = \pi^+(x)$ is a circle not equal to $\partial F([0, t], x)$, so $F([0, t], x)$ must be an annulus and one of the components of its boundary is equal to $\pi^+(x)$. ❑

PROPOSITION 3.9: Let $\lambda = \sup\{s: F(s, x) \text{ is a point}\}$. Denote by y the unique element of $F(\lambda, x)$. Then $D(x) = D(y)$ and for any negative trajectory $T(x)$ through x we have $D(x)\backslash T(x) = D(y)\backslash T(y)$, where $T(y) \cup \pi([0, \lambda], y) = T(x)$.

The proof follows immediately from the definitions of $D(x)$ and boundary trajectories.

4. MAIN RESULTS

According to Proposition 3.9 we may assume, without loss of generality, that x is a singular point. Thus, throughout this section x denotes a nonstationary singular point. Recall that by τ we denote the period of x if x is periodic.

LEMMA 4.1: Let γ be a Jordan curve contained in $D(x)$. Then there exists a $t > 0$ such that $\gamma \subset F([0, t), x)$.

Proof: Take a $p \in \gamma$. There is a t_p with $p \in F(t_p, x)$ and (by 3.6 and 3.7) $p \in \operatorname{Int} F([0, t_p + \frac{\tau}{2}], x)$. The family $\{\operatorname{Int} F([0, t_p + \frac{\tau}{2}, x) : p \in \gamma\}$ is an open covering of γ; by the compactness of γ we get $\gamma \subset \bigcup\{\operatorname{Int} F([0, t_i + \frac{\tau}{2}], x) : i = 1, \ldots, n\}$; if we fix $t = \tau + \max\{t_1, \ldots, t_n\}$, we obtain that $\gamma \subset F([0, t), x)$. ❑

LEMMA 4.2: If a singular point x is regular, we assume that t is an arbitrary positive number; if x is periodic, we assume that $t < \tau$. Then the set $F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$ is homeomorphic to $\mathbb{R}^2$.

This is a consequence of Proposition 3.7.

The following lemma is essential to the proof of our main theorems.

LEMMA 4.3: Assume that x is a periodic point and $t > \tau$. Then $F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$ is homeomorphic to $\mathbb{R}^2$.

Proof:

Case 1: Assume that $\pi^+(x)$ is not a boundary trajectory.

According to 3.7, $F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0,t], x)) = \operatorname{Int} F([0, t), x) \backslash J$, where $J = T_1([0, t-\tau], x) \cup T_2([0, t-\tau], x)$ is an arc as x is singular.

For each $s \in [0, \tau - t]$, consider the arc $\Gamma(s) = T_1([s, t-\tau], x) \cup F(s, x) \cup T_2([s, t-\tau], x)$. Note that $\Gamma(0) = J$, $\Gamma(t-\tau) = F(t-\tau, x)$, so $F(t-\tau, x)$ is deformed into J as s varies from $t - \tau$ to 0. The end-points of the arc $\Gamma(s)$ belong to $\partial F([0, t], x)$, the non-end-points of $\Gamma(s)$ belong to $\operatorname{Int} F([0, t), x)$.

Now it is enough to show that $\operatorname{Int} F([0,t), x) \backslash F(t-\tau, x)$ is homeomorphic to $\mathbb{R}^2$. Since $F([0, t], x)$ is an annulus or a Möbius strip, then either $\operatorname{Int} F([0, t), x) \backslash F(t - \tau, x)$ is homeomorphic to $\mathbb{R}^2$ or $\operatorname{Int} F([0,t), x) \backslash F(t-\tau, x)$ is not connected. We show that $\operatorname{Int} F([0,t), x) \backslash F(t-\tau, x)$ is connected which will finish the proof in Case 1. Indeed, let $y_1, y_2 \in \operatorname{Int} F([0, t), x) \backslash F(t-\tau, x) = \pi((0, \tau), F(t, x) \backslash \{a_t, b_t\})$ (a_t and b_t are the end-points of $F(t, x)$); then $\pi(s_1, y_1) = \pi(s_2, y_2) = x$ for some $s_1, s_2 \in (t-\tau, t)$. We may assume that $s_1 \le s_2$, so there exists a $z \in F(s_2, x)$ with $\pi(s_2 - s_1, z) = y_1$. Denote by K the subarc of $F(s_2, x)$ joining z with y_2. Thus $K \cup \pi([0, s_2 - s_1], t)$ is a connected set, contained in $\operatorname{Int} F([0,t), x) \backslash F(t-\tau, x)$, joining y_1 with y_2.

Case 2: Assume that $\pi^+(x)$ is a boundary trajectory.

According to Corollary 3.8 and Remark 3.5 we have that $F([0,t], x)$ is an annulus, $T_1([0,t],x) = \pi^+(x)$ and $T_2([t, t-\tau],x)$ is contained in $\partial F([0,t], x)$. Moreover, there is an $\alpha \in [0,\tau)$ such that $T_2([-\alpha, 0], x) \subset \pi^+(x)$ and $T_2([t-\tau, -\alpha], x)$ is an arc with the end-points belonging to different components of $\partial F([0,t], x)$, all non-end-points of this arc belong to $\operatorname{Int} F([0,t], x)$. This shows that $F([0,t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$ is homeomorphic to the plane and finishes the proof. ❑

We can now formulate the first main result of the paper.

THEOREM 4.4: The set $D(x)$ is homeomorphic to the plane.

Proof: We show that $D(x)$ is an open, connected and simply connected set, which according to the classification theorems [14, Theorem 3] proves that $D(x)$ is homeomorphic to $\mathbb{R}^2$.

Step 1: Take a $y \in D(x)$; therefore $y \in F(s, x) \backslash (T_1(x) \cup T_2(x))$ for some s. There is a t (if x is periodic, we take $t > \tau$) with $y \in F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$, the last set is contained in $D(x)$ and, according to 4.2 and 4.3, open.

Step 2: We show that $D(x)$ is arcwise connected. Take $y, z \in D(x)$, hence there are $s_1, s_2 > 0$ with $y \in F(s_1, x)$, $z \in F(s_2, x)$, $y, z \notin T_1(x) \cup T_2(x)$. Put $t = s_1 + s_2 + \tau$ (for a regular point x take $\tau = 1$). Then y and z belong to $F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$ which (again from 4.2 and 4.3) is an arcwise connected set contained in $D(x)$. Thus we can find an arc joining y and z, contained in $D(x)$.

Step 3: Take a Jordan curve $\gamma \subset D(x)$. From Lemma 4.1 we can find a t with $\gamma \subset F([0, t), x)$; if x is periodic we take $t > \tau$. Therefore we have $\gamma \subset F([0, t), x) \backslash (T_1([0, t], x) \cup T_2([0, t], x))$; according to 4.2 and 4.3 the last set is homeomorphic to $\mathbb{R}^2$ and contained in $D(x)$, which gives that γ is contractible in $D(x)$. This proves that $D(x)$ is simply connected. ❑

Before we turn to the Jordan curve type theorems we need the following

LEMMA 4.5: If a trajectory given by a nonboundary negative solution σ is not equal to $\pi^+(x)$, then $L_\sigma^-(x) \cap D(x) = \varnothing$.

Proof: First assume that σ is an injection. Suppose to the contrary that $p \in L_\sigma^-(x) \cap D(x)$. From 3.6 and 3.7 we have that $p \in \operatorname{Int} F([0, t], x)$ for some $t > 0$. By the definition of $L_\sigma^-(x)$ we can find a sequence $\{t_n\}$ with $t_n \to \infty$, $\pi(t_n, \sigma(-t_n)) = x$ and $\sigma(-t_n) \to p$; it follows that for sufficiently large n we have $\sigma(-t_n) \in F([0, t], x)$ and $t_n > t$. Then $x = \pi(t_n, \sigma(-t_n)) = \pi(s, \sigma(-t_n)) = \sigma(s - t_n)$ for some $s \in [0, t]$ and $s - t_n < 0$, which is a contradiction.

If σ is not an injection then there is a k such that $\sigma|_{(-\infty, -k\tau]}$ is an injection and $\sigma(-k\tau) = x$. Put $\sigma'(u) = \sigma(-k\tau + u)$ for $u \leq 0$; then σ' is an injective negative solution through x with $L_{\sigma'}^-(x) = L_\sigma^-(x)$ which finishes the proof. ❑

The next main theorems of the paper are Theorem 4.6 and Theorem 4.7. They will be proved jointly.

THEOREM 4.6: Let x be a regular point. Then for every nonboundary negative trajectory T through x the set $D(x) \backslash T$ has two components, each of them homeomorphic to $\mathbb{R}^2$.

THEOREM 4.7: Let x be a periodic point. Then for every nonboundary negative trajectory T through x the set $D(x) \backslash T$ has either two or three components, each of them homeomorphic to $\mathbb{R}^2$. The set $D(x) \backslash T$ has three components if and only if $\pi^+(x)$ is not a boundary trajectory and the trajectory T fulfills the condition (c) of Remark 3.5 (in other words, $\pi^+(x) \subset T$ and $\pi^+(x) \neq T$).

Proof:

Step 1: First assume that a trajectory T is given by an injective negative solution σ. Then there is a $\lambda \leq 0$ such that $\sigma[\lambda, 0]$ is contained in one of the boundary trajectories and $\sigma(-\infty, \lambda) \subset D(x)$. Applying Lemma 4.5 we get that $\sigma(u)$ has no cluster point in $D(x)$ whenever $u \to \lambda^-$ and $u \to -\infty$. Denote by S the one-point compactification of $D(x)$; $S = D(x) \cup \{\infty\}$ and S is homeomorphic to a 2-dimensional sphere. Consider $\sigma(-\infty, \lambda)$ as a subset of S. Thus $\sigma(u) \to \infty$ for $u \to \lambda^-$ and for $u \to -\infty$. Therefore, $\sigma(-\infty, \lambda) \cup \{\infty\}$ is a Jordan curve contained in S, so it cuts S into two components, each of them homeomorphic to $\mathbb{R}^2$. This shows that

$\sigma(-\infty, \lambda)$ also cuts $D(x) = S\backslash\{\infty\}$ into two regions, each one homeomorphic to $\mathbb{R}^2$. We have that $D(x)\backslash T = D(x)\backslash\sigma(-\infty,\lambda)$ which finishes the proof in the case of a trajectory given by an injective solution.

Step 2: Take $T = \pi^+(x)$ and assume that $\pi^+(x)$ is nonboundary trajectory. Denote by σ the unique negative solution with the range equal to the trajectory $\pi^+(x)$ (compare Remark 3.5). There are $\alpha_1, \alpha_2 \in [0, \tau)$ such that $\sigma(s) = \sigma_i^x(s)$ for $s \in [-\alpha_i, 0]$ and $\sigma(-\infty, -\alpha_i) \cap \sigma_i^x(-\infty, -\alpha_i) = \emptyset$, $i = 1, 2$. If we put $\alpha = \max\{\alpha_1, \alpha_2\}$, then $\pi^+(x) \cap D(x) = \sigma(-\tau, -\alpha) = \pi((0, \tau-\alpha), x)$. The last set is a homeomorphic image of an open interval $(0, \tau-\alpha)$ with no cluster point in $D(x)$ when $u \to 0^+$ and $u \to (\tau-\alpha)^-$. The same reasoning as in Step 1 implies that $D(x)\backslash\pi^+(x)$ has two components, each of them homeomorphic to $\mathbb{R}^2$.

Step 3: Assume that σ is a noninjective solution with $\sigma(-\infty, 0] = T$ and $\pi^+(x)$ is not a boundary trajectory. Hence, there is a $k \geq 1$ such that $\sigma(-k\tau) = x$ and $\sigma|_{(-\infty,-k\tau]}$ is an injection. Note that $\sigma[-k\tau, 0] = \pi^+(x)$. Using Step 2 we get that $D(x)\backslash\pi^+(x)$ has two components, say D_1 and D_2, and $\partial D_1 = \partial D_2 \subset \pi^+(x)$. We show that $\sigma(-\infty, -k\tau] \subset \mathrm{Cl}\,D_1$ or $\sigma(-\infty, -k\tau] \subset \mathrm{Cl}\,D_2$. Suppose, contrary to our claim, that $\sigma(\alpha_1) \in D_1$ and $\sigma(\alpha_2) \in D_2$ for some α_1, α_2, say $\alpha_1 < \alpha_2$. Therefore, $\sigma[\alpha_1, \alpha_2] \cap \pi^+(x) \neq \emptyset$, so $\sigma(\beta) \in \pi^+(x)$ for some $\beta \in (\alpha_1, \alpha_2)$, and consequently $\sigma(\alpha_2) = \pi(\alpha_2 - \beta, \sigma(\beta)) \in \pi^+(x)$ which contradicts the fact that $\sigma(\alpha_2) \in D_2$. We may assume without loss of generality that $\sigma(-\infty, -k\tau] \subset \mathrm{Cl}\,D_1$. According to Remark 3.5 we have that $\sigma(-\infty, s) \subset D_1$ and $\sigma(s, -k\tau] \subset \pi^+(x)$ for some $s \in (-k\tau - \tau, -k\tau]$. Now consider the injective solution σ' given by $\sigma'(u) = \sigma(-k\tau + u)$ for $u \leq 0$. The same reasoning as in Step 1 (applied to the case of the region D_1 and the solution σ') gives that $D_1\backslash T = D_1\backslash\sigma'(-\infty, 0]$ has two components, each of them homeomorphic to $\mathbb{R}^2$. Thus finally we obtain that $D(x)\backslash T$ has three components, each of them homeomorphic to the plane.

Step 4: Consider the case where $\pi^+(x)$ is a boundary trajectory and σ is not injective. As in Step 3 we have that $T = \sigma(-\infty, -k\tau] \cup \pi^+(x)$ and $\sigma|_{(-\infty,-k\tau]}$ is injective. Here we get $D(x)\backslash T = D(x)\backslash\sigma(-\infty, -k\tau]$ as $\pi^+(x) \cap D(x) = \emptyset$ and the theorem follows from Step 1. ❑

EXAMPLE 4.8: Let us define a system $(\mathbb{R}^2, \mathbb{R}_+, \pi)$ as follows (in polar coordinates):

$$\pi(t, (r, \varphi)) = \begin{cases} 0 & \text{if } r = 0, \\ (1, \varphi + t) & \text{if } r = 1 \\ (r + t, \varphi) & \text{if } 0 < r + t \leq 1, \\ (1, \varphi + r + t - 1) & \text{if } r + t \geq 1 \text{ and } r < 1 \\ (r - t, \varphi) & \text{if } r - t \geq 1 \\ (1, \varphi + t - r - 1) & \text{if } r - t \leq 1 \text{ and } r > 1. \end{cases}$$

Consider $x = (1, 0)$. Then $F(x) = \mathbb{R}^2\backslash\{(0, 0)\}$; the boundary trajectories are equal to $(0,1] \times \{0\}$ and $[1, \infty) \times \{0\}$. Consider the negative trajectories $T_1 = (\{1\} \times [-\frac{\pi}{4}, 0]) \cup ([1, \infty) \times \{-\frac{\pi}{4}\})$ and $T_2 = (\{1\} \times [-\frac{9\pi}{4}, 0]) \cup ([1, \infty) \times \{-\frac{\pi}{4}\})$. Then $D(x)\backslash T_1$ has two components and $D(x)\backslash T_2$ has three components. Some trajectories of the system are shown in Figure 3.

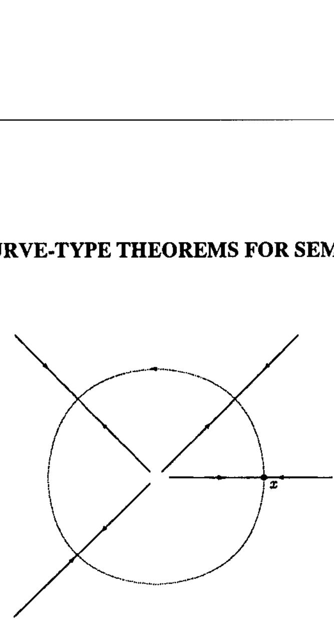

FIGURE 3

REMARK 4.9: According to [5, 4.6] we get that the same description of $F(x)$ and $D(x)$ remains valid for a semiflow on an arbitrary metric space (not necessarily 2-manifold) under the assumption that x is a nonstationary point and (possibly after a suitable isomorphism, see [11]) there is a $t_0 \geq 0$ with $F(t, x)$ being a point for $t \leq t_0$ and an arc for $t \geq t_0$.

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On MAD Families and Sequential Order

ALAN DOW[a]

Department of Mathematics
York University
North York, Ontario M3J 1P3

ABSTRACT: We investigate the question of the existence of a compact sequential space with sequential order more than 2. We establish a connection with an interesting problem about maximal families of almost disjoint subsets of ω. We also obtain a partial result by showing that it is consistent that every countably tight compact space of weight b is Fréchet-Urysohn. The main task is to work with Shelah's forcing for obtaining a model of $b < s = a$.

1. INTRODUCTION

The sequential order of a space is the supremum of the number of times that one must iterate the taking limits of convergent sequences in order to get the closure; that is, for $A \subset X$, $A^{(0)} = A$, $A^{(1)} = \{x \in X : (\exists \{x_n : n \in \omega\} \subset A)\ \{x_n : n \in \omega\}$ converges to $x\}$, and $A^{(\alpha)} = (\bigcup_{\beta < \alpha} A^{(\beta)})^{(1)}$. A space is sequential if and only if it has a sequential order ($\leq \omega_1$), and a space is Fréchet-Urysohn if its sequential order is one. A simple example of a compact space with sequential order two is the one-point compactification of a Ψ-space, i.e., a space defined from a maximal almost disjoint family of a subsets of ω.

It was shown, by Bashkirov [4], to follow from CH that there are compact spaces of any sequential order up to ω_1 and that MA implies the existence of a compact space of sequential order three.

2. TOTALLY MAD FAMILIES

Let us consider a simple construction of a compact space with sequential order three from the assumption $b = c$. We will then introduce an interesting property and question about mad families arising from this construction (what we will call totally mad families).

PROPOSITION 2.1: ($b = c$) There is a compact space with sequential order 3.

Proof: We define two families of subsets of ω, $\{a_\alpha : \alpha < c\}$ and $\{b_\alpha : \alpha < c\}$, by induction on α. The inductive hypotheses are

(1) $\{a_\beta : \beta < \alpha\}$ is an almost disjoint family,
(2) for each $\beta, \gamma < \alpha$, a_β is either almost contained in b_γ or is almost disjoint from it,

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[a]E-mail: Alan.Dow@mathstat.yorku.ca.

(3) for $\beta < \gamma < \alpha$, there is a finite subset (possibly empty) $F_{\beta,\gamma}$ of $\gamma + 1$ such that $b_\beta \cap b_\gamma$ is almost equal to $\bigcup\{a_\xi : \xi \in F_{\beta,\gamma}\}$.

The goal is to have that $\{a_\alpha : \alpha < c\}$ is a maximal almost disjoint family and for each infinite $A \subset c$ there is an infinite $A' \subset A$ and a $\beta < c$ so that $a_\alpha \subset^* b_\beta$ for each $\alpha \in A'$. Therefore, we simply fix enumerations $\{x_\alpha : \alpha < c\}$ and $\{y_\alpha : \omega \leq \alpha < c\}$ of $\mathcal{P}(\omega)$ and $[c]^\omega$, respectively, where $y_\alpha \subset \alpha$ for all $\alpha \geq \omega$.

So we add two more conditions to our inductive hyptheses:

(4) each $\beta < \alpha$, x_β meets some member of $\{a_\gamma : \gamma \leq \beta\}$ in an infinite set,

(5) each $\gamma < \alpha$, there is $\beta \leq \gamma$ so that $a_\xi \subset^* b_\beta$ for infinitely many ξ in y_γ.

Let us show that we can find suitable a_α and b_α. Consider x_α; if $x_\alpha \cap a_\beta$ is infinite for some $\beta < \alpha$ then let $a_\alpha = \emptyset$. So assume that x_α does not meet a_β in an infinite set for any $\beta < \alpha$. If there is a $\beta < \alpha$ such that x_α meets b_β in an infinite set, then let $a_\alpha = x_\alpha \cap b_\beta$ (since x_α is almost disjoint from all the previously chosen a_γ's, we have, by inductive hypothesis 3, there is at most one such β!). If there is no such β, then we can let $a_\alpha = x_\alpha$.

Next consider the sequence $\{a_\xi : \xi \in y_\alpha\}$. If there is a $\beta < \alpha$ such that $a_\xi \cap b_\beta$ is infinite (hence, cofinite in a_ξ) for infinitely many $\xi \in y_\alpha$, then let $b_\alpha = \emptyset$. Otherwise note that since $b = c$, we can find a function $h \in {}^{y_\alpha}\omega$ such that for each $\beta < \alpha$, there is a finite $F_{\beta,\alpha} \subset y_\alpha$ such that $a_\xi \subset^* b_\beta$ for each $\xi \in F_{\beta,\alpha}$, $a_\xi \cap b_\beta$ is finite for each $\xi \in y_\alpha - F_{\beta,\alpha}$ and $(a_\xi - h(\xi)) \cap b_\beta$ is empty for all but finitely many $\xi \in y_\alpha$. Furthermore, $a_\beta \cap \bigcup_{\xi \in y_\alpha} a_\xi - h(\xi)$ is finite for all $\beta \leq \alpha$, $\beta \notin y_\alpha$. So we let $b_\alpha = \bigcup_{\xi \in y_\alpha} a_\xi - h(\xi)$. Note that it would suffice if we could do this for an infinite subset of y_α.

The Stone space of the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by the union of the two families $\{a_\alpha : \alpha < c\}$ and $\{b_\alpha : \alpha < c\}$ together with the finite sets is our desired space X. The usual Ψ-space constructed from the maximal almost disjoint family $\{a_\alpha : \alpha < c\}$ is a subspace of X. The two scattering levels of Ψ constitute the first two scattering levels of X. The next scattering level of X consists of ultrafilters corresponding to each b_α, i.e., $\{b_\alpha - (\bigcup F \cup n) : n \in \omega$ and F is a finite subset of $\{a_\gamma : \gamma < c\}\}$ generates an ultrafilter of B. There is only one other point in the space, i.e., we have the one-point compactification of the points described so far. It is therefore trivial that X is compact and sequential. The inductive hypotheses 4 and 5 guarantee that the sequential order of X is at least 3, in fact it is equal to 3. ❑

In the above construction, consider the step in which we chose b_α. We were given the family $\{a_\xi : \xi \in y_\alpha\}$ (an arbitrary countable subfamily of $\{a_\xi : \xi < c\}$). We can fix an enumeration $\{\xi_n : n \in \omega\}$ of y_α and, by removing a finite set from each, we may assume that the a_{ξ_n} are pairwise disjoint. Now, for each γ, there is a partial function f_γ from a subset of ω to ω such that $a_\gamma \cap a_{\xi_n}$ is not empty iff n is in the domain of f_γ and, for n in the domain of f_γ, $f_\gamma(n)$ is the largest element of $a_\gamma \cap a_{\xi_n}$. Similarly, we may assume that $b_\gamma \cap a_{\xi_n}$ is either empty or infinite for each n not in the domain of f_γ' and, for other n, $f_\gamma'(n)$ is the largest value of $a_{\xi_n} \cap b_\gamma$. In the above proof, we found a function h which bounded all of the f_γ and f_γ' for $\gamma < \alpha$. However, it was not necessary to have a total function; it would have sufficed to pass to an infinite subset of $\{\xi_n : n \in \omega\}$. We would say that $\{a_\gamma : \gamma < \alpha\}$ is not totally unbounded with respect to $\{a_{\xi_n} : n \in \omega\}$ as per the following definition.

DEFINITION 2.2: (1) Given $\mathcal{B}$ and $\mathcal{A}$, infinite families of subsets of ω, say that $\mathcal{A}$ is *totally unbounded* with respect to $\mathcal{B}$, if for each infinite $\mathcal{B}' \subset \mathcal{B}$ and each $h \in {}^{\mathcal{B}'}\omega$, there is an $a \in \mathcal{A}$ such that $a \cap \bigcup\{b - h(b) : b \in \mathcal{B}'\}$ is infinite.

(2) A maximal almost disjoint family $\mathcal{A}$ is *totally* mad if for each infinite $\mathcal{B} \subset \mathcal{A}$ no subset of $\mathcal{A}$ of cardinality less than c is totally unbounded with respect to $\mathcal{B}$.

QUESTION 1: (1) Is there a totally mad family?
(2) Does $b = \omega_1$ imply there is a totally mad family?

REMARK 2.1: A totally mad family has a refinement which is a completely separable mad family. A mad family is said to be completely separable (introduced by Hechler and Shelah [5], [7]) if each $X \subset \omega$ which meets infinitely many in an infinite set will actually contain one of them. The existence of a completely separable mad family has been established under various cardinal hypotheses [3], but it is not known to follow from $b = \omega_1 < s$. Our hope is that that there is a modification of Shelah's model of $b < s$ (discussed later in the paper) in which every compact separable sequential space will have sequential order at most 2 and that in all likelihood there will be no totally mad family.

The next result will be an integral tool in our construction of a totally mad family (Proposition 2.6).

PROPOSITION 2.3: Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω into infinite parts. Then there is a family $\mathcal{D}$ such that;
(1) $\mathcal{R} \cup \mathcal{D}$ is a mad family on ω;
(2) every $D \in \mathcal{D}$ is a transversal of $\mathcal{R}$, i.e., $|D \cap R_n| \le 1$ for every $n \in \omega$;
(3) whenever $\mathcal{B} \in [\mathcal{R}]^\omega$ and $\mathcal{D}_0 \in [\mathcal{D}]^{<c}$, then there is an infinite $\mathcal{B}' \subseteq \mathcal{B}$ and a function $g : \mathcal{B}' \to \omega$ such that the set $D \cap \bigcup\{B \backslash g(B) : B \in \mathcal{B}'\}$ is finite for every $D \in \mathcal{D}_0$.

Therefore, the family $\mathcal{R} \cup \mathcal{D}$ is a mad family in which no subset of $\mathcal{D}$ with cardinality less than c is totally unbounded with respect to $\mathcal{R}$.

Proof: By [2], we may fix $\mathcal{T}$, a tree π-base for $[\omega]^\omega$ of height at most b (i.e., $\mathcal{T} \subset [\omega]^\omega$ forms a tree under the ordering of reverse inclusion mod finite and $\mathcal{T}$ is dense in $[\omega]^\omega$ with respect to this ordering). It is easily arranged that $\mathcal{T}$ is dense in $[\omega]^\omega$ with respect to the usual reverse inclusion ordering since we can just ensure that for each $T \in \mathcal{T}$ and each $n \in \omega$, there is a $T' \in \mathcal{T}$ with $T' \subset T - n$. For $X \subseteq \omega$, let $\operatorname{dom} X$ denote the set $\{n \in \omega : X \cap R_n \neq \emptyset\}$. For every $X \subseteq \omega$ with $\operatorname{dom} X$ infinite, choose some $T(X) \in \mathcal{T}$ so that $T(X) \subseteq \operatorname{dom} X$ and $T(X) \neq T(X')$, whenever $X \neq X'$. This is clearly possible since for every infinite $M \subseteq \omega$, the set of all $T \in \mathcal{T}$ with $T \subseteq M$ is of size c.

Enumerate $[\omega]^\omega$ as $\{X_\alpha : \alpha < c\}$ and proceed by transfinite induction. Suppose we have found $\{D_\beta : \beta < \alpha\}$ for some $\alpha < c$. If X_α is almost disjoint with all R_n, $n \in \omega$, and all D_β, $\beta < \alpha$, then let D_α be a subset of X_α, which is a transversal of $\mathcal{R}$ and such that $\operatorname{dom} D_\alpha = T(X_\alpha)$, otherwise let $D_\alpha = \emptyset$. Let $\mathcal{D} = \{D_\alpha : \alpha < c, D_\alpha \neq \emptyset\}$.

The conditions (1) and (2) immediately follow from the inductive construction. In order to verify (3), let $\mathcal{D}_0 \in [\mathcal{D}]^{<c}$ and $\mathcal{B} \in [\mathcal{R}]^\omega$. There is some $T_0 \in \mathcal{T}$

such that for every $n \in T_0$, $R_n \in \mathcal{B}$. There are c almost disjoint members of $\mathcal{T}$, all contained in T_0 and $|\mathcal{D}_0| < c$; hence, there is some $T \subseteq T_0$, $T \in \mathcal{T}$ such that the inclusion $\operatorname{dom} D \subseteq^* T$ holds for no $D \in \mathcal{D}_0$. Since $\mathcal{T}$ is a tree under $^*\supseteq$ and since $\operatorname{dom} D \in \mathcal{T}$ for every $D \in \mathcal{D}$, we have, for $D \in \mathcal{D}_0$, that either $T \cap \operatorname{dom} D$ is finite or $T \subseteq^* \operatorname{dom} D$. For $D \in \mathcal{D}_0$, let $f_D \in {}^T\omega$ be defined by $f_D(n) = 0$, if $n \notin \operatorname{dom} D$, and by $f_D(n) = k$, if $n \in T$ and $\{k\} = D \cap R_n$. Since the height of $\mathcal{T}$ is less or equal to b, there is some $f \in {}^T\omega$ with $f \;{}^*\!> f_D$ for all $D \in \mathcal{D}_0$ such that $\operatorname{dom} D \;{}^*\!\supseteq T$. Now, it remains to put $\mathcal{B}' = \{R_n : n \in T\}$, define $g(R_n) = f(n)$ and (3) holds with this choice of $\mathcal{B}'$ and g.

COROLLARY 2.4: Let $\alpha \in \omega_1$ be a limit ordinal, let $C = \{C_\beta : \beta < \alpha\}$ be a decreasing mod finite chain of subsets of ω. Then there is a family $\mathcal{D}(C) \subseteq [\omega]^\omega$ such that:

(1) for every $D \in \mathcal{D}(C)$ and every $\beta < \alpha$, $D \subseteq^* C_\beta$;
(2) $\mathcal{D}(C)$ is an almost disjoint family;
(3) $\mathcal{D}(C)$ is maximal with respect to (1) and (2);
(4) if $\mathcal{B}$ is an almost disjoint family of sets such that for every $B \in \mathcal{B}$ there is some $\beta < \alpha$ with $B \subseteq^* (C_\beta \backslash C_{\beta+1})$ and if $\mathcal{D}_0 \in [\mathcal{D}(C)]^{<c}$, then there is some infinite $\mathcal{B}' \subseteq \mathcal{B}$ and a function $g: \mathcal{B}' \longrightarrow \omega$ such that the set
$$D \cap \bigcup\{B \backslash g(B) : B \in \mathcal{B}'\}$$
is finite for every $D \in \mathcal{D}_0$.

Proof: Choose an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ cofinal in α. Let $R_0 = \omega - C_{\alpha_0}$ and for $n > 0$ define $R_n = C_{\alpha_n} \backslash (C_{\alpha_n + 1} \cup \bigcup_{k<n} R_k)$. Apply Proposition 2.3 to get $\mathcal{D} = \mathcal{D}(C)$. (1), (2), and (3) of the corollary follow from (1) and (2) of Proposition 2.3. If there is some $\beta < \alpha$ such that the set $\mathcal{B}' = \{B \in \mathcal{B} : B \cap C_\beta \text{ is finite}\}$ is infinite, put g to be the constant 0 and (4) follows. Otherwise there is some infinite set $\mathcal{B}'$ such that the set $\{\beta < \alpha : B \subseteq^* C_\beta, B \in \mathcal{B}'\}$ is cofinal in α and moreover, for every $n \in \omega$, there is at most one $B \in \mathcal{B}'$ with $B \subseteq^* R_n$. Now, Proposition 2.3 easily applies. ❑

LEMMA 2.5: Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω with each R_n infinite, let $Y \in [\omega]^\omega$ be almost disjoint with all R_n. Then there is a family $\mathcal{H} = \{H_\alpha : 0 < \alpha < b\}$ such that:

(1) $H_{\alpha+1}$ (for each $\alpha < b$) is an infinite set, which is almost disjoint with R_n for each $n \in \omega$;
(2) H_α (for each limit $\alpha < b$) is a decreasing mod finite chain, $H_\alpha = \{C_{\alpha,\beta} : \beta < \alpha\}$ with each $C_{\alpha,\beta}$ almost disjoint with R_n for each $n < \omega$;
(3) if $\alpha < \beta < b$, then $H_{\alpha+1} \cap H_{\beta+1}$ is finite;
(4) if $\alpha < \beta \leq \gamma < \delta < b$, then $C_{\beta,\alpha} \cap C_{\delta,\gamma}$ is finite;
(5) if $\alpha < \beta < \gamma < b$ or $\beta < \gamma \leq \alpha < b$, then $H_{\alpha+1} \cap C_{\gamma,\beta}$ is finite;
(6) whenever $X \in [\omega]^\omega$ is almost disjoint with each R_n ($n \in \omega$), then either $X \cap H_{\alpha+1}$ is infinite for some $\alpha < b$ or there is some limit $\alpha < b$ such that $X \cap C_{\alpha,\beta}$ is infinite for all $\beta < \alpha$;
(7) $|Y \cap H_1| = \omega$.

Proof: Enumerate $R_n = \{r(n,k) : k \in \omega\}$ and fix a family of strictly increasing functions $\{f_\alpha : \alpha < b\} \subset {}^\omega\omega$ which is unbounded with respect to $<^*$ and, moreover,

$f_\alpha <^* f_\beta$ for $\alpha < \beta < b$. In addition, assume that $\{n : Y \cap \{r(n, k) : k < f_1(n)\} \neq \varnothing\}$ is infinite. Let $H_1 = \{r(n, k) : n \in \omega,\ k \in \omega,\ k < f_1(n)\}$; $H_{\alpha+1} = \{r(n, k) : n \in \omega,\ k \in \omega,\ f_\alpha(n) \leq k < f_{\alpha+1}(n)\}$ for $0 < \alpha < b$, and finally let $C_{\alpha,\beta} = \{r(n,k) : n \in \omega,\ k \in \omega,\ f_\beta(n) \leq k < f_\alpha(n)\}$ for $\beta < \alpha < b$, α limit.

It is straightforward to verify that, since for all $\alpha < \beta < b, f_\alpha(n) < f_\beta(n)$ holds for almost all $n \in \omega$, (1) up to (5) are satisfied. (6) holds because the family $\{f_\alpha : \alpha < b\}$ is unbounded and all the functions f_α are strictly increasing. By the choice of f_1, (7) holds. ❑

PROPOSITON 2.6: ($\mathfrak{b} = s = \omega_1$ or $\mathfrak{b} = c$). There is a totally mad family on ω.

Proof: Of course the statement follows trivially from $b = c$, so we assume $b = s = \omega_1$. Fix a splitting family $\{S_\xi : \xi < \omega_1\}$ and an unbounded family of functions $\{f_\alpha : \alpha < b\} \subseteq {}^\omega\omega$ satisfying: for $\alpha < \beta < b$, $f_\alpha <^* f_\beta$, moreover, $\lim f_\beta(n) - f_\alpha(n) = \infty$ and $f_0 \geq id$.

We shall construct simultaneously, for $\xi < \omega_1$, a family $\mathfrak{W}_\xi \subset [\omega]^\omega$ consisting of decreasing mod finite chains of countable limit length, a mad family $\mathcal{A}_\xi$ on ω and an almost disjoint family $\mathcal{D}_\xi$, with the aim to get $\bigcup_{\xi<\omega_1}\mathcal{D}_\xi$ totally mad.

The starting point is quite simple: let $\mathcal{D}_0 = \{D_n : n \in \omega\}$ be an arbitrary partition of ω with each D_n infinite, let $\mathfrak{W}_0$ consist of just one chain C, where $C = \{\omega \setminus \bigcup_{k<n} D_k : n \in \omega\}$ and let $\mathcal{A}_0$ be the union of $\mathcal{D}_0 \cup \mathcal{D}(C)$, where $\mathcal{D}(C)$ is the family described in Corollary 2.4.

Let $\xi < \omega_1$ and suppose that all $\mathcal{D}_\eta$, $\mathfrak{W}_\eta$, and $\mathcal{A}_\eta$ are known for $\eta < \xi$. The induction assumptions are as follows:

(a) for every $\eta < \zeta < \xi$, $\mathcal{A}_\zeta$ refines $\mathcal{A}_\eta$ and $\bigcup_{\xi \leq \zeta}\mathcal{D}_\xi \subset \mathcal{A}_\zeta$;

(b) for every $\eta < \zeta < \xi$ and for every chain $C \in \mathfrak{W}_\eta$ there is some $C' \in \mathfrak{W}_\zeta$ with $C \subseteq C'$ and C' is an end-extension of C;

(c) for every $\eta < \xi$, for every $D \in \bigcup_{\zeta \leq \eta}\mathcal{D}_\zeta$ and for every chain $C \in \mathfrak{W}_\eta$ there is some $C \in C$ with $D \cap C$ finite, similarly, for two distinct chains $C, C' \in \mathfrak{W}_\eta$ there is $C \in C$ and $C' \in C'$ with $C \cap C'$ finite;

(d) for every $\eta < \xi$ and for every chain $C \in \mathfrak{W}_\eta$ there is a family $\mathcal{D}(C)$ chosen as in Corollary 2.4, such that for each $A \in \mathcal{A}_\eta$ either there is a $C \in C$ such that $A \cap C$ is finite, or there is a $D \in \mathcal{D}(C)$ such that $A \subset D$;

(e) for every $\eta < \xi$ and for every infinite $X \subseteq \omega$, either $|X \cap D| = \omega$ for some $D \in \bigcup_{\zeta \leq \eta}\mathcal{D}_\zeta$ or there is a chain $C \in \mathfrak{W}_\eta$ such that $|X \cap C| = \omega$ for all $C \in C$;

(f) if $\eta + 1 < \xi$, then for every $C \in \mathfrak{W}_{\eta+1}$ there is some $C \in C$ with $C \subseteq S_\eta$ or $C \cap S_\eta = \varnothing$;

(g) if $\eta + 1 < \xi$ and $C \in \mathfrak{W}_\eta$, then for every infinite $\mathcal{B} \subset \mathcal{D}_{\eta+1}$ such that, for each $D \in \mathcal{B}$, $D \subset^* C$ for all $C \in C$, there is a $C' \in \mathfrak{W}_{\eta+1}$ and a descending mod finite sequence $\{C_n : n \in \omega\} \subset C'$ such that, for each n, there is a $B \in \mathcal{B}$ which is contained mod finite in $C_n - C_{n+1}$ while if $\eta < \xi$ is a limit ordinal, then $\mathcal{D}_\eta$ is empty.

If $\xi < \omega_1$ is limit, let $\mathfrak{W}_\xi$ consist of all chains C such that for every $\eta < \xi$ there is some $C_\eta \in \mathfrak{W}_\eta$ with $C = \bigcup_{\eta<\xi} C_\xi$. For every $C \in \mathfrak{W}_\xi$, let $\mathcal{D}(C)$ be the almost disjoint family resulting from the application of Corollary 2.4 to C. Then $\mathcal{B}$ defined as $\bigcup_{\eta<\xi}\mathcal{D}_\eta \cup \bigcup\{\mathcal{D}(C) : C \in \mathfrak{W}_\xi\}$ is a maximal almost disjoint family on ω; let $\mathcal{A}_\xi$ be an arbitrary mad family on ω which refines $\mathcal{B}$ as well as all $\mathcal{A}_\eta$, $\eta < \xi$.

Since $\xi < \omega_1$, such an $\mathcal{A}_\xi$ is easy to find. Finally, put $\mathcal{D}_\xi = \emptyset$.

For a successor ordinal $\xi = \eta + 1$, we proceed more carefully. First, denote by $\mathcal{Y}$ the family of all $Y \in [\omega]^\omega$ such that the set $\{C \in \mathfrak{M}_\eta$: for every $C \in C$, $|Y \cap C| = \omega\}$ is of size c. Then for every $Y \in \mathcal{Y}$ select one chain $C(Y) \in \mathfrak{M}_\eta$ such that Y meets all $C \in C(Y)$ in an infinite set and do it so that distinct Y's are paired to distinct $C(Y)$'s.

Consider $C \in \mathfrak{M}_\eta$. There is some $\alpha < \omega_1$ such that $C = \{C_\beta : \beta < \alpha\}$ and in the previous induction step some cofinal sequence $\langle \alpha_n : n \in \omega \rangle$ was selected in order to find $\mathcal{D}(C)$ using Corollary 2.4. Let $R_n = C_{\alpha_n} \backslash (C_{\alpha_{n+1}} \cup \bigcup_{k<n} R_k)$ and apply Lemma 2.5 to the family $\mathcal{R} = \{R_n : n \in \omega\}$ to select the family $\{H_\alpha : \alpha < b\}$. Since all members of $\mathcal{D}(C)$ are transversals of $\mathcal{R}$ and since $\lim f_{\alpha+1}(n) - f_\alpha(n) = \infty$, every $H_{\alpha+1}$ meets uncountably many D's from $\mathcal{D}(C)$, hence, also uncountably many A's from $\mathcal{A}_\eta$. For every $\alpha < b$ ($= \omega_1$) choose a countable family $\{A_n(C, \alpha) : n \in \omega\} \subseteq \mathcal{A}_\eta$ such that $A_n(C, \alpha) \cap H_{\alpha+1}$ is infinite for every n, and put $D_n(C, \alpha) = \{A_n(C, \alpha) \cap H_{\alpha+1} : n \in \omega\}$. In the case that $C = C(Y)$, just one modification is necessary: we shall guarantee that $D_0(C, 1)$ meets Y in an infinite set. This is always possible with the aid of Lemma 2.5(7), and due to the fact that $\mathcal{A}_\eta$ is a mad family. Let $\mathcal{D}_\xi$ be the set of all infinite members of $\{D_n(C, \alpha) \cap S_\eta : n \in \omega, \alpha \in \omega_1, C \in \mathfrak{M}_\eta\} \cup \{D_n(C, \alpha) - S_\eta : n \in \omega, \alpha \in \omega_1, C \in \mathfrak{M}_\eta\}$.

If $C \in \mathfrak{M}_\eta$, then $\mathfrak{M}_\xi(C)$ will denote the family, which we now define, of all chains to which C extends. We now take care of condition (f). For $\alpha + 1 < \omega_1$, if all sets $H_{\alpha+1} \cap S_\eta \backslash \bigcup_{k<n} D_k(C, \alpha)$ are infinite, then $C \cup \{H_{\alpha+1} \cap S_\eta \backslash \bigcup_{k<n} D_k(C, \alpha) : n \in \omega\}$ belongs to $\mathfrak{M}_\xi(C)$. Similarly, if all sets $H_{\alpha+1} \cap (\omega \backslash S_\eta) \backslash \bigcup_{k<n} D_k(C, \alpha)$ are infinite, then $C \cup \{H_{\alpha+1} \cap (\omega \backslash S_\eta) \backslash \bigcup_{k<n} D_k(C, \alpha) : n \in \omega\}$ belongs to $\mathfrak{M}_\xi(C)$ as well. If $\alpha < \omega_1$ is limit, then $C \cup \{H_\alpha\}$ is a chain, however, we again take the member S_η from the splitting family into account: $C \cup \{C\alpha, \beta \cap S_\eta : \beta < \alpha\}$ and $C \cup \{C_{\alpha,\beta} \backslash S_\eta : \beta < \alpha\}$ will be members of $\mathfrak{M}_\xi(C)$, provided they do not contain finite sets. Clearly, for each $0 < \alpha < \omega_1$, at least one of the chains defined above is put into $\mathfrak{M}(C)$. Set $\mathfrak{M}_\xi = \bigcup \{\mathfrak{M}_\xi(C) : C \in \mathfrak{M}_\eta\}$.

It remains to define $\mathcal{A}_\xi$. Similar to the limit step, let $\mathcal{D}(C)$ be the almost disjoint family resulting from the application of Corollary 2.4 to the chain C. Then $\mathcal{B} = \bigcup_{\eta \le \xi} \mathcal{D}_\eta \cup \bigcup \{\mathcal{D}(C) : C \in \mathfrak{M}_\xi\}$ is a mad family and it remains to define $\mathcal{A}_\xi = \{A \cap B : A \in \mathcal{A}_\eta, B \in \mathcal{B}, |A \cap B| = \omega\}$.

This completes the inductive definitions. The verification of the induction assumptions is straightforward and will be left to the reader.

We have to verify that $\mathcal{D} = \bigcup_{\xi < \omega_1} \mathcal{D}_\xi$ is totally mad. The almost disjointness of $\mathcal{D}$ should be clear, so let us first prove its maximality. Let $Y \in [\omega]^\omega$ be arbitrary, assume that $\xi(0) < \omega_1$ is the first ordinal such that both sets $Y \cap S_{\xi(0)}$ and $Y \backslash S_{\xi(0)}$ are infinite. If Y does not meet any $D \in \bigcup_{\eta \le \xi(0)} \mathcal{D}_\eta$ in an infinite set, then by (e), there is some chain C in $\mathfrak{M}_\xi$ such that $Y \cap C$ is infinite for all $C \in C$, however, by (f), there must be two chains C_0 and C_1 in $\mathfrak{M}_\xi$ with this property. Choose two infinite subsets Y_0 and Y_1 of Y such that $Y_0 \subseteq^* C$ for all $C \in C_0$ and $Y_1 \subseteq^* C$ for all $C \in C_1$. Let $\xi(1)$ be maximum of the first ζ with both $S_\zeta \cap Y_0$ and $Y_0 \backslash S_\zeta$ infinite and of the first η with both $S_\eta \cap Y_1$ and $Y_1 \backslash S_\eta$ infinite. If Y still meets no $D \in \bigcup_{\eta \le \xi(1)} \mathcal{D}_\eta$ in an infinite set, then Y_0 is compatible with two chains C_{00}, $C_{01} \in \mathfrak{M}_{\xi(1)}$, both extending C_0, and similarly for $Y(1)$. Proceeding further, if the branching continues, we finally arrive at a limit $\xi = \sup \{\xi(n) : n \in \omega\}$. Since Y

meets all members from c chains from $\mathfrak{M}_\eta$, our construction ensures that $Y \in \mathcal{Y}$ at the $(\xi+1)$-*st* step of the induction and so $Y \cap D$ is infinite for some $D \in \mathcal{D}_{\xi+1}$. Since Y was arbitrary, we conclude that $\mathcal{D}$ is a mad family.

Let $\mathcal{B} \in [\mathcal{D}]^\omega$, $\mathcal{E} \in [\mathcal{D}]^{<c}$ be two disjoint subsets of $\mathcal{D}$. Then there is some infinite $\{B_n : n \in \omega\} \subseteq \mathcal{B}$, some $\xi < \omega_1$ and a chain $C = \{C_\beta : \beta < \alpha\} \in \mathfrak{M}_\xi$ such that for some increasing sequence of ordinals $\langle \beta_n : n \in \omega \rangle$ cofinal in α, $B_n \subseteq^* C_{\beta_n} \backslash C_{\beta_{n+1}}$. To see this, let δ be minimal such that $\mathcal{B} \cap \bigcup_{\eta<\delta} \mathcal{D}_\eta$ is infinite; hence, we may assume that $\mathcal{B} \subset \bigcup_{\eta<\delta}\mathcal{D}_\eta$. If $\delta = 0$ then it is clear that we may take C to be the unique member of $\mathfrak{M}_0$. Otherwise, recursively choose a sequence $C_\eta \in \mathfrak{M}_\eta$ so that $\beta < \eta$ implies $C_\beta \subset C_\eta \in \mathfrak{M}_\eta$ and there are infinitely many $B \in \mathcal{B}$ which are contained mod finite in every member of C_η. Clearly there is some $\xi \le \delta$ for which we cannot choose C_ξ. If $\xi = \beta + 1$ is a successor, then we can apply inductive hypothesis (g) to find the required $C \in \mathfrak{M}_\xi$ and sequence $\langle \beta_n : n \in \omega \rangle$. If on the other hand, ξ is a limit, then $C = \bigcup_{\eta<\xi} C_\eta \in \mathfrak{M}_\xi$ is the desired chain and it should be clear, by inductive hypotheses (a) and (d), that there is a sequence $\langle \beta_n : n \in \omega \rangle$ as required (i.e., there is an infinite sequence $\langle B_n : n \in \omega \rangle \subset \mathcal{B}$ and an increasing sequence $\langle \eta_n : n \in \omega \rangle \subset \xi$ such that B_n is almost contained in every member of C_{η_n} but is almost disjoint from some member of C_{η_n+1}).

Having found C, let α be the length of C and let $\langle \alpha_n : n \in \omega \rangle$ be the increasing sequence that was used in the application of Corollary 2.4 when defining $\mathcal{D}(C)$. Choose an infinite subset $\mathcal{B}_0$ of $\mathcal{B}$ so that for each $n \in \omega$, there is at most one $B \in \mathcal{B}_0$ such that $B \subset^* C_{\alpha_n} - C_{\alpha_{n+1}}$. Now, by construction, we have that for every $E \in \mathcal{E}$, either there is an n such that $E \cap C_{\alpha_n}$ is finite or there is a $D \in \mathcal{D}(C)$ such that $E \subset^* D$ (by hypotheses (a) and (d)). Therefore, by Lemma 2.3, there is a suitable $\mathcal{B}' \subset \mathcal{B}$ and function $h \in {}^{\mathcal{B}'}\omega$ witnessing that $\mathcal{E}$ is not totally unbounded with respect to $\mathcal{B}$. The proposition is proved. ❑

3. SHELAH'S $b < a$ POSET

For the reader's convenience we will reproduce the results from [9] that we will need in the next section.

DEFINITION 3.1.1: Let K_n be the family of pairs (s,h), s a hereditarily finite set, h a partial function from $\mathcal{P}(s)$ to $n+1$ such that

(1) $h(s) = n$

(2) if $h(t) = \ell + 1$ $(t \subseteq s)$, $t = t_1 \cup t_2$ then $h(t_1) \ge \ell$ or $h(t_2) \ge \ell$.

DEFINITION 3.1.2: Let $K_{\ge n} = \bigcup_{k \ge n} K_k$ and $K = \bigcup_n K_n$.

One example of a member of $(s,h) \in K_n$ is to take s to be the set of integers $[0, 2^n]$ and to define $h(t) = \log_2(|t|)$ for $t \subseteq s$.

DEFINITION 3.2: An ordering on K is defined in two steps:

(1) Suppose (s,h) and (s',h') are members of K, we say that $(s,h) \le^d (s',h')$ (we think of (s,h) as refining (s',h')) if:
 (a) $s = s'$;
 (b) $\mathrm{dom}(h) \subset \mathrm{dom}(h')$;
 (c) for some integer i, $h(u) = \max\{0, h'(u) - i\}$ for all $u \in \mathrm{dom}(h)$.

(2) $(s,h) \leq^e (s',h')$ if for some $\bar{s} \in \operatorname{dom} h'$, $(s,h) = (\bar{s}, h' \restriction \mathcal{P}(\bar{s}))$.
(3) $(s, h) < (s', h')$ if for some $(\bar{s}, \bar{h})$,

$$(s, h) \leq^d (\bar{s}, \bar{h}) \leq^e (s', h').$$

We next define a set L consisting of finite trees of members of K whose maximal elements will be integers. These will form the component pieces of the conditions of our poset and they represent a certain measure of how likely it is that the finite sets of integers will be contained in the generic filter.

DEFINITION 3.3: L is the union of all the L_n where for each n, L_n is the family of pairs (S, H) such that:
(1) S is a finite tree with a root;
(2) $\operatorname{int}(S)$ is the set of maximal nodes of S and will be a subset of ω, the set of nonmaximal nodes of S will be denoted $\operatorname{in}(S)$ and we will assume that $\operatorname{in}(S)$ is disjoint from ω;
(3) H is a function whose domain is $\operatorname{in}(S)$ and let H_x denote $H(x)$ for $x \in \operatorname{in}(S)$;
(4) For $x \in \operatorname{in}(S)$, $(\operatorname{Suc}_S(x), H_x) \in K_{\geq n}$ where $\operatorname{Suc}_S(x)$ is the set of immediate successors of x in S.

If $(S, H) \in L$, then $\operatorname{lev}(S, H) = \max\{n : (S, H) \in L_n\}$.

The ordering on K will lift naturally to an ordering on L as follows.

DEFINITION 3.4: We say that $(S^0, H^0) \leq (S^1, H^1)$ if $S^0 \subseteq S^1$, they have the same root, $\operatorname{in}(S^0) = S^0 \cap \operatorname{in}(S^1)$ and, for every $x \in \operatorname{in}(S^0)$, $(\operatorname{Suc}_{S^0}(x), H_x{}^0) \leq (\operatorname{Suc}_{S^1}(x), H_x{}^1)$.

If $t \in L$, we use (S^t, H^t) to denote its components. For $(S, H) \in L$ and $x \in \operatorname{in}(S)$, let $(S, H)^{[x]} = (S^{[x]}, H \restriction S^{[x]})$ where $S^{[x]} = \{y \in S : x \leq y \text{ in } S\}$. An important tool in the proofs is to define $\operatorname{half}(S, H)$ for (S, H) in L_n, where $\operatorname{half}(S, H) = (S', H')$ is defined by $S' = S$, $\operatorname{dom}(H_x') = \{A \in \operatorname{dom}(H_x) : H_x(A) \geq n/2\}$ and $H_x'(A) = [H_x(A) - n/2]$ (the greatest integer function). The usefulness is in the following.

FACT 1: If $t \in L_n$ and $t' \leq \operatorname{half}(t)$, then $\bar{t}$ (canonically obtained from t' and t) is such that $\bar{t} \leq t$, $\operatorname{int}(t') = \operatorname{int}(t)$, and $\bar{t} \in L_{[n/2]}$. Define $\bar{t}$ as follows: $S^{\bar{t}} = S^{t'}$ and $H^{\bar{t}} = \mathrm{H}^t \restriction S^{t'}$ in the sense that for each $x \in S^{\bar{t}}$, $H_x{}^{\bar{t}} = \mathrm{H}^t{}_x \restriction \mathcal{P}(\operatorname{Suc}_{S^{t'}}(x))$.

An important fact for getting the splitting number to go up, as well as in some of the proofs of the properties of the poset we will define is the following.

FACT 2: If $(S, H) \in L_{n+1}$ and $\operatorname{int}(S) = A_0 \cup A_1$, then there is $(S', H') \leq (S, H)$ such that $\operatorname{int}(S') \subseteq A_0$ or $\operatorname{int}(S') \subseteq A_1$ and $(S', H') \in L_n$.

DEFINITION 3.5: The forcing-notion Q is defined as follows:
(1) $p \in Q$ if $p = (W, T)$ where W is a finite subset of ω and T is a countably infinite set of pairwise disjoint members of L such that $T - L_n$ is finite for each n. Let $\operatorname{int}(p)$ and $\operatorname{int}(T)$ denote the set $\bigcup \{\operatorname{int}(t) : t \in T\}$.
(2) Given $t_1 = (S_1, H_1), \ldots, t_k = (S_k, H_k)$ all from L such that $S_i \cap S_j = \varnothing$ for $i \neq j$, t is *built* from $t_1, \ldots, t_k$ if: There are incomparable nodes $a_1, \ldots, a_k$ of S such that every node of S is comparable with one of the a_i and such that $(S, H)^{[a_i]} = (S_i, H_i)$ for $i = 1, \ldots, k$.

(3) $(W,T) \leq (W',T')$ if $W' \subset W \subset W' \cup \operatorname{int}(T')$ and letting $T' = \{t'_0, t'_1, \ldots\}$ and $T = \{t_0, t_1, \ldots\}$, there are $\bar{t}_i \leq t'_i$ for each i and pairwise disjoint finite subsets, $B_0, B_1, \ldots$, of $\{\bar{t}_i : i \in \omega\}$ such that, for each $m \in \omega$, t_m is built from B_m. It follows that for each n only finitely many of the B_m's will meet $L - L_n$.

(4) We also demand that for $(W,T) \in Q$ and $t, t' \in T$, $\max W < \min(\operatorname{int}(t))$ and either $\max(\operatorname{int}(t)) < \min(\operatorname{int}(t'))$ or vice-versa.

It is natural to think of W as the root of the condition $(W,T) \in Q$ and root preserving extensions will be important. The reader familiar with Axiom A forcings will observe that Q is an Axiom A poset and there is a natural choice for $<_n$ similar to the $<_0$ defined next.

DEFINITION 3.6: For $p \in Q$ we use (W^p, T^p) and $p <_0 q$ will mean that $p < q$ and $W^p = W^q$. In addition, we will use $\{t_n{}^p : n \in \omega\}$ to denote T^p listed so that $\max(\operatorname{int}(t_n))$ is increasing. If $p \in Q$, $n \in \omega$, and $W \subseteq 1 + \max(\operatorname{int}(t_n{}^p))$ then $p_W{}^n$ denotes the condition $(W, \{t_\ell{}^p : \ell > n\})$.

DEFINITION 3.7: For a proper ideal $I \subset \mathcal{P}(\omega)$ (which includes all finite sets) let $Q[I]$ denote the set of $q \in Q$ such that for each $A \in I$, there are infinitely many $t \in T^q$ such that $\operatorname{int}(t) \cap A$ is empty.

PROPOSITION 3.8: (1) $p \in Q$ and τ_n are Q-names of ordinals, then there is a $q <_0 p$, such that for each $k \leq n < \omega$ and each $W \subseteq \max[\operatorname{int}(t_n{}^q)] + 1$, $q_W{}^n$ forces a value on τ_k iff some $<_0$ extension of $q_W{}^n$ forces a value on τ_k.

(2) Q is proper.

(3) $\Vdash_Q$ "$\{n : (\exists p \in G_Q)\ [\, n \in W^p]\}$ is an infinite subset of ω which $\mathcal{P}(\omega)^V$ does not split."

Proof: Parts (1) and (2) are proven below for $Q[I]$, the more difficult case, so we will not prove them here. Item (3) follows from Fact 2 above. ❑

PROPOSITION 3.9: Let C_{ω_1} denote the poset for adding ω_1 Cohen reals. In $V^{C_{\omega_1}}$ the following hold for an ideal $I \in V$ with $[\omega]^{<\omega} \subset I \subset \mathcal{P}(\omega)$.

(1) If $p \in Q[I]$ and τ_n are $Q[I]$-names of ordinals then there is a $<_0$ extension q of p such that: $q \in Q[I]$, and for each $k \leq n < \omega$ and each $W \subset \max[\operatorname{int}(t_n)] + 1$, $q_W{}^n$ forces a value on τ_k iff some $<_0$ extension of $q_W{}^n$ forces a value on τ_k.

(2) $Q[I]$ is proper.

(3) $\Vdash_{Q[I]}$ "$\{n : (\exists p \in G_{Q[I]})\ n \in W^p\}$ is an infinite subset of ω which is almost disjoint from every $A \in I$."

Proof: (1) Let λ be a large enough regular cardinal and let M be a countable elementary submodel of $(H(\lambda), \in, V \cap H(\lambda))$ to which all of I, the set of ω_1 many Cohen reals given by C_{ω_1}, $Q[I]$, p, and τ_n, for each $n \in \omega$, all belong. Let $\delta = M \cap \omega_1$. Define by induction on $n < \omega$, $q^n \in Q[I] \cap M$, t_n and $k_n < \omega$ such that:

(1) each q^n is a $<_0$ extension of p;

(2) $q^n \leq q^\ell$ for $\ell < n$ and if $W \subset k_n$, $m < n+1$ and some $<_0$ extension of (W, T^{q^n}) forces a value on τ_m, then (W, T^{q^n}) does so;

(3) $k_n > k_\ell$ and $k_n > \max(\operatorname{int}(t_\ell))$ for $\ell < n$;

(4) every $\ell \in \operatorname{int}(q^n)$ is greater than k_n;

(5) $t_n \in T^{q^n}$ and $\mathrm{lev}(t_n) > n$ and $\min(\mathrm{int}(t_n))$ is greater than k_n.

To do so, first choose k_n, then q^n, and at last t_n. We want in the end to let $T^q = \{t_n : n < \omega\}$. One point is missing. Why does $q = (W^p, T^q)$ belong to $Q[I]$ (not just to Q)? The answer is that it may not but we may choose a real $r \in {}^{\omega}\omega$ which is Cohen generic over $V[M]$ and choose t_n to be the $r(n)$-th possible one according to some fixed well-ordering from V of the hereditarily finite sets (i.e., all possible t's). Then for any $A \in I$ there are infinitely many such t_n for which $\mathrm{int}(t_n) \cap A = \varnothing$, hence, there will be infinitely many n such that $\mathrm{int}(t_n) \cap A$ is empty.

Now the proof of (2) is quite standard. Let M etc. be as above. Recall that from [8], it suffices to show that there is a condition $q < p$ such that for each $\tau \in M$ which is a $Q[I]$-name of an ordinal, q forces that τ takes on a value from M. So in condition (1) we may take $\{\tau_n : n \in \omega\}$ to be the list of all $Q[I]$-names of ordinals which are members of M. Let r be any extension of q which forces a value on some τ_k. Let $n > k$ be chosen so that $W^r \subset \max[\mathrm{int}(t_n)] + 1$. Clearly $(W^r, \{t_l : l > n\})$ has a $<_0$-extension (namely r) which forces a value on τ_k. Therefore, $q^n_{W^r}$ forces a value on τ_k and, since $q_{W^r}{}^n$ is a member of M, this value is in M.

Statement (3) follows trivially from the fact that for each $A \in I$, the set $\{q \in Q[I] : \mathrm{int}(T^q) \cap A = \varnothing\}$ is dense in $Q[I]$. ❑

We will also need the fact that Q is almost ${}^{\omega}\omega$-bounding — this is certainly the property of Q which is the most difficult to obtain (and which does not hold for $Q[I]$ for many choices of I). This proof we will need to reproduce carefully as we will need to generalize it in the next section. We refer the reader to [8] for the basic properties of almost ${}^{\omega}\omega$-bounding forcings: e.g., that a countable support iteration of almost ${}^{\omega}\omega$-bounding proper posets is itself weakly bounding. We give the definitions below.

LEMMA 3.10: Let $(\varnothing, T) \in Q$ and let $\mathcal{W}$ be a family of finite subsets of $\mathrm{int}(T)$ so that

$$\text{for every } (\varnothing, T') < (\varnothing, T), \text{ there is a } W \in \mathcal{W} \text{ with } W \subset \mathrm{int}(T'). \tag{3.1}$$

Then for each $k \in \omega$, there is a $t \in L_k$ appearing in some $(\varnothing, T') < (\varnothing, T)$ such that, for every $t' \le t$, there is a $W \in \mathcal{W}$ with $W \subseteq \mathrm{int}(t')$.

Proof: Let T be enumerated as $\{t_n : n \in \omega\}$ and assume with no loss of generality that $\mathcal{W}$ is closed upwards. We first do the case $k = 1$. The desired tree t will be built from a set $\{t_n' : n \in u\}$ where $t_n' \le t_n$ for each n. For this we need a finite subset u of ω together with a partial function $H : \mathcal{P}(u) \to \omega$ such that $H(u) = 1$ and $H(u_1) \ge 0$ or $H(u_2) \ge 0$ whenever $u = u_1 \cup u_2$ and whenever $H(v) \ge 0$, then for every choice $\bar{t}_n \le t_n' \, (n \in v)$ the set $\bigcup_{n \in v} \mathrm{int}(t_n)$ belongs to $\mathcal{W}$.

This suggests defining a partial function $H : [\omega]^{<\omega} \to \omega$ by the following conditions:

1. $u \in \mathrm{dom}(H)$ iff for every $l \in u$ the tree $\mathrm{half}(t_l)$ belongs to L_1 and for all possible choices $t_l' \le \mathrm{half}(t_l)$ $(l \in u)$ the set $\bigcup_{l \in u} \mathrm{int}(t_l')$ belongs to $\mathcal{W}$, and
2. $H(u) \ge m + 1$ iff whenever $u = u_1 \cup u_2$ one has $H(u_1) \ge m$ or $H(u_2) \ge m$.

It is assumed tacitly that $H(u)$ is always chosen as large as possible so that for example, $H(u) = 0$ implies there are $v, w \subset u$ such that $u = v \cup w$ but $v, w \notin \mathrm{dom}(H)$. The need to use $\mathrm{half}(t_\ell)$ will become apparent shortly.

Once we have u with $H(u) \geq 1$ we define $t = (S^t, H^t)$ by taking u as the root of the tree S^t and putting the trees $\mathrm{half}(t_\ell)$ as its immediate successors. The function $H_u{}^t$ is taken to be H on the successors of u and the appropriate $H^{\mathrm{half}(t_\ell)}$ higher up. This defines an element of L_1 that is built using trees that were below elements of T and as such it appears in a condition $(\varnothing, T')$ below $(\varnothing, T)$.

Now if $t' < t$ then the set of successors of u that are in t' forms a set u' that is in $\mathrm{dom}(H^{t'})$ and hence in $\mathrm{dom}(H)$ so that we have $H(u') \geq 0$. But t' is built using $t_\ell' \leq \mathrm{half}(t_\ell)$ with $\ell \in u'$, so $\mathrm{int}(t') = \bigcup_{\ell \in u'} \mathrm{int}(t_\ell')$ belongs to $\mathcal{W}$.

We show that there is such a u in three stages.

Stage A: There is n such that if $t_\ell' < \mathrm{half}(t_\ell)$ is chosen for each $\ell < n$, then $\bigcup_{\ell<n} \mathrm{int}(t_\ell') \in \mathcal{W}$. This is because, otherwise, the family of all $\langle t_\ell' : \ell < n\rangle$ such that $n < \omega$, $t_\ell' < \mathrm{half}(t_\ell)$ for $\ell < n$ and $\bigcup_{\ell<n} \mathrm{int}(t_\ell') \notin \mathcal{W}$ forms an ω-tree with finite branching. By König's lemma there is an infinite branch, say $\langle t_n' : n \in \omega\rangle$. But now there is a T' such that $(\varnothing, T') < (\varnothing, T)$ and $\mathrm{int}(T') = \bigcup_{n\in\omega} \mathrm{int}(t_n')$. The only problem with taking $T' \subset \{t_n' : n \in \omega\}$ is that $\{\mathrm{lev}(t_n') : n \in \omega\}$ may be bounded. However, by Fact 1, we can replace each t_n' by some $\bar{t}_n$ where the level of $\bar{t}_n$ is at least one-half of that of t_n. It follows that $T' = \{\bar{t}_n : n \in \omega\}$ is a suitable choice. Of course, the hypothesis on $\mathcal{W}$ contradicts the assumption that $\langle t_n' : n \in \omega\rangle$ is a branch through the above tree — hence we finish. Note that if $A \subset \omega$ is such that $\{n : \mathrm{int}(t_n) \cap A = \varnothing\}$ is infinite, then this is true of T' as well. Therefore, this stage generalizes directly for $Q[I]$.

Stage B: Repeated applications of Stage A will give $0 = n(0) < n(1) < \cdots$, such that for every i and choice of $t_\ell' < \mathrm{half}(t_\ell)$ $(n(i) \leq \ell < n(i+1))$, the set $\bigcup\{\mathrm{int}(t_\ell') : n(i) \leq \ell < n(i+1)\}$ is an element of $\mathcal{W}$. Note that if i is large enough then $\mathrm{half}(t_\ell)$ belongs to L_1 for all $\ell \geq n(i)$, so that $\big[n(i), n(i+1)\big) \in \mathrm{dom}(H)$ for all but finitely many i.

Stage C: Similarly we can produce a sequence $0 = m(0) < m(1) < \cdots$ such that for every $j \in \omega$ and function $h : \big[m(j), m(j+1)\big) \to \omega$ with $h(i) \in \big[n(i), n(i+1)\big)$ for all $i \in \mathrm{dom}(h)$ and for every choice $t_{h(i)}' \leq \mathrm{half}(t_{h(i)})$, the set $\bigcup\{\mathrm{int}(t_{h(i)}') : m(j) \leq i < m(j+1)\}$ belongs to $\mathcal{W}$. Let us remark, however, that if A is some subset of ω for which there are infinitely many n such that $\mathrm{int}(t_n)$ is disjoint from A, then this may not be true for a subsequence of t_n's. Therefore, this stage does not easily generalize to $Q[I]$.

Now, take j large enough so that $\big[n(i), n(i+1)\big) \in \mathrm{dom}(H)$ for $i \geq m(j)$ and consider $u = \big[n(m(j)), n(m(j+1))\big)$; we claim that $H(u) \geq 1$. Indeed, let $v \subseteq u$. If there is i such that $[n(i), n(i+1)) \subset v$, then $v \in \mathrm{dom}(H)$, otherwise we may define $h : [m(j), m(j+1)) \to u$ with $h(i) \in [n(i), n(i+1)) \backslash v$ for all i. It then follows, by Stage C, that $u \backslash v \in \mathrm{dom}(H)$.

Now observe that the above arguments apply to all $(\varnothing, T') \leq (\varnothing, T)$ and hence that the family $\mathcal{W}_1$ consisting of those W that contain sets of the form $\bigcup_{\ell\in u} \mathrm{int}(t_\ell')$, where $H(u) \geq 1$ and $t_\ell' \leq \mathrm{half}(t_\ell)$ for all ℓ, also satisfies (3.1). We may define its associated function H_1. We note that $H_1(u) \geq 0$ implies $H(u) \geq 1$ and so

$H_1(u) \geq 1$ implies $H(u) \geq 2$. Using high enough t_ℓ we can then find our required $t \in L_2$. An obvious induction completes the proof. ❑

We can now reformulate this lemma in terms of forcing.

LEMMA 3.11: Fix $p \in Q$, a sequence $\{\tau_n : n \in \omega\}$ of Q-names of ordinals, and let $q = (W^q, \{t_n : n \in \omega\})$ be as in Proposition 3.8, then there is an $r <_0 q$ such that if $T^r = \{\bar{t}_n : n \in \omega\}$ the following holds.

> For every $n < \omega$, $W \subseteq \left[0, \max(\text{int}(t_n)) + 1\right)$ and $t''_{n+1} \leq \bar{t}_{n+1}$, there is $W' \subset \text{int}(t''_{n+1})$ such that $(W \cup W', \{\bar{t}_\ell : \ell > n+1\})$ forces a value on $\tau_m (m \leq n)$. (3.2)

Proof: Choose, inductively $\bar{t}_n$ using Lemma 3.10 as follows. Having chosen $\bar{t}_{n-1}$, let m be large enough so that $\max[\text{int}(\bar{t}_{n-1})] < \min[\text{int}(t_m)]$. Let $\mathcal{W}$ be the set of all $W \subset [\min[\text{int}(t_m)], \omega)$ such that for each W' with $W^q \subset W' \subset \max[\text{int}(t_{n-1})] + 1$ there is an $r \leq (W', \{t_\ell : m \leq \ell < \omega\})$ such that $W^r - W' \subseteq W$ and r forces a value on each of τ_k for $k \leq n$. Clearly $\mathcal{W}$ satisfies the conditions in Lemma 3.10. Apply Lemma 3.10 to $\{t_\ell : \ell \geq m\}$ to obtain a condition $\bar{t}_n \in L_n$. ❑

Recall that a poset Q is *almost* ${}^\omega\omega$-bounding if for every Q-name of a function from ω to ω and $p \in Q$, there is some $g \in {}^\omega\omega$ such that for every infinite $A \subset \omega$, there is a condition $p' < p$ such that

$$p' \Vdash_Q \{n \in A : f(n) < g(n)\} \text{ is infinite.}$$

A poset is *weakly bounding* if every unbounded family of functions from ω to ω remains unbounded in the extension.

PROPOSITION 3.12: The forcing Q is almost ${}^\omega\omega$-bounding.

Proof: Let $\dot{f}$ be a Q-name of a function from ω to ω and let $p \in Q$. For each n, let $\tau_n = \dot{f}(n)$ and obtain $r = (W^r, T^r)$ as in Lemma 3.11. Let $g : \omega \to \omega$ be the function where $g(n)$ is the largest k such that for some $W \subset \max(\text{int}(t^r{}_{n+1})) + 1$, the condition $(W, \{t_\ell{}^r : \ell > n+1\}$ forces that $\dot{f}(n) = k$. Given any infinite $A \subset \omega$, let $p_A = (W^p, \{t^r_{n+1} : n \in A\})$. Let $\tilde{r} \leq p_A$ and let $n \in A$ be such that some $\tilde{t} \leq t_n{}^r$ is used in building some member of $T^{\tilde{r}}$. Of course $W^{\tilde{r}}$ is one of the subsets used in defining $g(n)$ and by Lemma 3.8, it follows that the value $\tilde{r}$ forces on τ_n is the same as that forced by $(W^{\tilde{r}}, \{t_\ell{}^r : \ell > n+1\})$. This completes the proof. ❑

4. THE MODIFICATION OF $Q[I]$

One thing that was not proven in the previous sections is that $Q[I]$ is almost ${}^\omega\omega$-bounding. Indeed, in general, we essentially only expect $C_{\omega_1} * Q[I]$ to be almost ${}^\omega\omega$-bounding and not even for all ideals I. In [9], Shelah shows that this is the case when I is the ideal generated by an almost disjoint family of subsets of ω. We can describe this another way. Consider the usual Ψ-space (so-called) generated by an almost disjoint family $\mathcal{A}$, i.e., the space is $\omega \cup \{x_A : A \in \mathcal{A}\}$ and a neighborhood base for x_A is $\{\{x_A\} \cup A - n : n \in \omega\}$. Let X be the one-point compactification of this space and let x denote the new point added. What we want to remark is that X is a compact sequential space and that I is equal to $I_x = \{A \subset \omega : x \notin \mathrm{cl}_X(A)\}$. In general, if X is a sequential space which is a compactification of ω and $x \in X$, let $I_{X,x} = \{A \subset \omega : x \notin \mathrm{cl}_X(A)\}$. We intend to show in this section that there is a forcing P which is a countable support iteration of proper almost ${}^{\omega}\omega$-bounding forcings so that $Q[I_{X,x}]$ is almost ${}^{\omega}\omega$-bounding in V^P (and so can be used in an iteration which will be weakly bounding).

We begin with the following. As usual, if X is a compact space and P is any forcing notion, then X generates canonically a compact space in the extension V^P as follows: assume that X is a subset of $[0,1]^{\kappa}$ for $\kappa = w(X)$. In the extension, we let X refer to the closure of X^V in $[0,1]^{\kappa}$.

LEMMA 4.1: (CH) Let X be a compactification of ω and let Q_{ω_1} be the countable support ω_1-length iteration in which Q is the iterand for each $\alpha < \omega_1$. Let $P = Q_{\omega_1} * C_{\omega_1}$. In V^P, if $(\emptyset, \{t_n : n \in \omega\}) \in Q$ then, there is an infinite $J \subset \omega$ and a sequence $t_n' \leq t_n$ for $n \in J$ so that $(\emptyset, \{t_n' : n \in J\}) \in Q$ and there is an $x \in X$ so that $\bigcup\{\mathrm{int}(t_n') : n \in J\}$ converges to x in the space X.

Proof: It should be clear that we may assume that $\delta < \omega_1$ is large enough so that $\{t_n : n \in \omega\}$ is a member of the model $V^{Q_\delta * C_\delta}$ where, of course, Q_δ is the iteration of δ copies of Q and C_δ is the obvious countable subposet of C_{ω_1}. Working in the model V^{Q_δ} we can inductively choose t_n' in such a way that for each forcing condition $p \in C_\delta$ there are infinitely many n such that there is a $p' < p$ which forces t_n to equal t_n' and such that $(\emptyset, \{t_n' : n \in \omega\})$ is a member of Q (i.e., $t_m' \in L - L_n$ for at most finitely many m). Now find a coordinate $\alpha > \delta$ so that $(\emptyset, \{t_n' : n \in \omega\})$ is a member of $G(\alpha)$ (i.e., the generic subset of Q given by the α-th coordinate in Q_{ω_1}). It is easy to check that, by genericity, $J_1 = \{n : (\exists \bar{t}_n)\, \bar{t}_n \leq t_n'$ and $(\exists q \in G(\alpha))\ \mathrm{int}(\bar{t}_n) \subset W^q\}$ is infinite. In addition we may assume (by passing to a subset) that $\mathrm{lev}(\bar{t}_n)$ diverges to infinity. For each $n \in J_1$, fix $\bar{t}_n \leq t_n'$ such that there is a $q \in G(\alpha)$ with $\mathrm{int}(\bar{t}_n) \subset W^q$. Finally, by the inductive construction of the t_n''s, there is an infinite set $J \subset J_1$ in V^P such that $t_n = t_n'$ for each $n \in J$. This is our desired set J. Note that $(\emptyset, \{\bar{t}_n : n \in J\})$ is a member of Q. The only thing remaining is to see that $\bigcup\{\mathrm{int}(\bar{t}_n) : n \in \omega\}$ converges to some member of X. This, however, follows from the fact (Lemma 3.8) that $\{m : (\exists q \in G(\alpha))\ m \in W^q\}$ is not split by any basic open subset of X. ❑

The key property that seems to make this go through is the following.

DEFINITION 4.2: A poset $P = Q[I_{X,x}]$ will be said to have the *madf property* if for each $p \in P$ with $T^p = \{t_n : n \in \omega\}$ there are pairwise disjoint subsets of ω, $\{J_n : n \in \omega\}$ and a sequence $\{t_n' : n \in \omega\}$ with $t_n' \leq t_n$ for each n, so that for each $B \subset \omega$ such that $B \cap J_n$ is infinite for each n, $(\emptyset, \{t'_n : n \in B\})$ is a member of P.

LEMMA 4.3: Suppose that X is a compactification of ω with countable tightness and suppose that for each $p \in Q$, there is a $y \in X$ and $p' \leq p$ such that $\mathrm{int}(T^{p'})$ converges to y. Then $Q[I_{X,x}]$ has the madf property for each $x \in X$.

Proof: Let S be the set of $y \in X$ such that there is a $J_y \subset \omega$ such that there is a $T_y \leq T_{J_y}$ for which $\mathrm{int}(T_y)$ converges to y (where $T_J = \{t_n : n \in J\}$). Note that by the hypothesis on X, every infinite subset of ω contains some such J. Since $p \in Q[I_{X,x}]$, x is in the closure of S. Since X has countable tightness, there is a countable subset $\{s_n : n \in \omega\} \subset S$ which has x as a limit point. Note that if $x \in S$, then we may stop since it would follow that, below $(\varnothing, T_x)$, the posets $Q[I_{X,x}]$ and Q are the same.

For each n, fix J_{s_n} and $\{t_m{}^n : m \in J_{s_n}\}$ so that $t_m{}^n \leq t_m$ for each m, the set $\bigcup\{\mathrm{int}(t_m{}^n) : m \in J_{s_n}\}$ converges to s_n and $(\varnothing, \{t_m{}^n : m \in J_{s_n}\})$ is in Q. Since we may shrink J_{s_n}, we may assume that $\{J_{s_n} : n \in \omega\}$ are pairwise disjoint (countable families of infinite sets have disjoint refinements). With no loss of generality (and it isn't important) we may assume that $\bigcup_{n \in \omega} J_{s_n}$ is all of ω. For each $m \in \omega$, there is a unique n, such that $m \in J_{s_n}$, define $t_m{}'$ to be $t_m{}^n$. Observe that $(\varnothing, \{t_m{}' : m \in \omega\})$ is a member of $Q[I_{X,x}]$. Now suppose $B \subset \omega$ is such that $B \cap J_{s_n}$ is infinite for each n, then $(\varnothing, \{t'_m : m \in B\})$ is a member of $Q[I_{X,x}]$. To see this, fix any $A \in I_{X,x}$. Since $x \notin \mathrm{cl}_X(A)$, there is an n such that $s_n \notin \mathrm{cl}_X(A)$. It follows that the intersection of A with $\bigcup\{\mathrm{int}(t_m{}^n) : m \in J_{s_n}\}$ is finite. Therefore, there are infinitely many members, m, of $B \cap J_{s_n}$ such that $\mathrm{int}(t_m{}^n) \cap A$ is empty. ❑

LEMMA 4.4: Suppose that X is a compactification of ω and suppose that $x \in X$ is such that $Q[I_{X,x}]$ has the madf property. Let $p = (\varnothing, T) \in Q[I_{X,x}]$ and suppose that $\mathcal{W}$ is a set of finite subsets of ω such that for each $(\varnothing, T') < (\varnothing, T)$, there is a $W \in \mathcal{W}$ with $W \subset \mathrm{int}(T')$. Then, for every $k \in \omega$, there is a $t \in L_k$ appearing in some $(\varnothing, T') < (\varnothing, T)$ such that, for every $t' \leq t$, there is a $W \in \mathcal{W}$ with $W \subset \mathrm{int}(t')$.

Proof: Let $T = \{t_n : n \in \omega\}$ and proceed as in Lemma 3.10. As mentioned in the proof of Lemma 3.10, the proofs of Stage A and Stage B are unchanged. We generalize the idea from [9], where this is shown for the case X is the one-point compactification of a Ψ-space, to prove Stage C. The rest of the proof is as in Lemma 3.10.

Fix the sequences $\{t_n{}' : n \in \omega\}$ and $\{J_n : n \in \omega\}$ as in Definition 4.2. The rest of the proof is virtually the same as in [9].

In Stage A, we may assume that $n(i+1)$ is chosen large enough so that $[n(i), n(i+1)) \cap J_k$ is not empty for each $k \leq n(i)$. Now fix a function ϕ from ω to ω so that $\phi(i) \leq n(i)$ for each i and so that $\phi^{-1}(n)$ is infinite for each n (ϕ can be defined by induction).

Stage C: We wish to define the sequence $m(\ell)$ so that for each function h with domain $[m(\ell), m(\ell+1))$ and with $h(i) \in [n(i), n(i+1)) \cap J_{\phi(i)}$ for each relevant i, and for each selection of $t_{h(i)} \leq \mathrm{half}(t_{h(i)}{}')$, the set $\bigcup \mathrm{int}(\bar{t}_{h(i)})$ is in $\mathcal{W}$. As in the proof of Stage A in Lemma 3.10, the idea is to note that if $m(\ell+1)$ cannot be chosen, then we obtain an infinite sequence $\{\bar{t}_{h(i)} : m(\ell) \leq i < \omega\}$ such that $h(i) \in [n(i), n(i+1)) \cap J_{\phi(i)}$ for each i and so that $\bar{t}_{h(i)} \leq \mathrm{half}(t_{h(i)}{}')$. The role of the $\mathrm{half}(t_n{}')$'s is the same in that they guarantee that we can find a suitable $(\varnothing, \overline{T}) \leq (\varnothing, T)$ in Q which does not contain a member of $\mathcal{W}$. However we must ensure that this $(\varnothing, \overline{T})$ is a member of $Q[I_{X,x}]$ to obtain a contradiction. We do obtain such a contradiction, since B, the range of h, meets each J_n in an infinite set. Therefore, just as above, there are, for each $A \in I_{X,x}$ infinitely many i so that $\bar{t}_{h(i)}$ misses A.

COROLLARY 4.5: (CH) If X is a compactification of ω and if, in $V^{C_{\omega_1}}$, x is a member of X such that $Q[I_{X,x}]$ has the madf property, then in $V^{C_{\omega_1}}$, $Q[I_{X,x}]$ is almost ${}^{\omega}\omega$-bounding.

Proof: We work in $V^{C_{\omega_1}}$ and let $\dot{f}$ be a $Q[I_{X,x}]$-name of an increasing function from ω to ω. Fix $\delta < \omega_1$ large enough so that $\dot{f} \in V^{C_\delta}$. We may invoke Lemmas 4.1 and 4.4, to find a sequence $\{\bar{t}_n : n \in \omega\}$ just as we did in Lemma 3.11 except that we may use a real which is Cohen generic over V^{C_δ} to choose the next value for $\bar{t}_n$ from among all those satisfying the requirements. This genericity will ensure that $(\emptyset, \{\bar{t}_n : n \in \omega\}) \in Q[I_{X,x}]$. We define a g similar to that defined in Lemma 3.12. First of all, since there is no harm in refining the sequence of $\bar{t}_n$'s, we may assume that there are pairwise disjoint J_n's so that for each B which meets each J_n in an infinite set, $(\emptyset, \{\bar{t}_n : n \in B\})$ is in $Q[I_{X,x}]$. Choose an increasing sequence $n(0,0) < n(0,1) < n(1,1) < n(0,2) < \cdots < n(0,\ell) < \cdots < n(\ell, \ell) \cdots$ so that $n(m, \ell) \in J_m$ for each $m \le \ell < \omega$. Clearly $(\emptyset, \{t(n(m, \ell)) : m \le \ell < \omega\})$ is in $Q[I_{X,x}]$. For each ℓ, define $g(\ell)$ large enough so that for each $W \subset \min \operatorname{int}(t(n(0,\ell)))$ and each $t \le \bar{t}(m, \ell)$ $(m \le \ell)$, there is a $W' \subset \operatorname{int}(t)$ such that $(W \cup W', \{\bar{t}_n : n(\ell, \ell) < n\})$ forces a value on $\dot{f}(n(m,\ell))$ which is less than $g(\ell)$. We may assume that g is increasing.

Fix any $A \in V^{C_{\omega_1}}$, and use the condition $p_A = (W^p, \{\bar{t}_{n(m,\ell)} : m \le \ell \in A\})$. To finish the proof, suppose that r is any condition below p_A. There is an $m < \ell \in A$ such that $\bar{t}_{(m,\ell)}$ is used in building a condition t' of T^r. We know that $\operatorname{int}(t')$ contains a set W' so that $(W^r \cup W', \{\bar{t}_n : n > n(\ell,\ell)\})$ forces a value on $\dot{f}(n(m, \ell))$ which is less than $g(\ell)$. Since $\dot{f}$ is increasing, it follows that $g(\ell)$ is forced to be greater than $\dot{f}(\ell)$. ❑

The payoff in this section is the following interesting observation.

THEOREM 4.6: There is a model of $b = \omega_1 < \omega_2 = s = a = c$ in which every compact space of countable tightness and weight ω_1 is actually Fréchet-Urysohn.

Proof: Start with a model of CH, and force with the countable support iteration of posets of the form $Q_{\omega_1} * C_{\omega_1} * Q[I_{X,x}]$ where a suitable diamond sequence is used to choose which X and x to use. More precisely, diamond will predict a space X containing ω and a point $x \in X$. If X has countable tightness after forcing with $Q_{\omega_1} * C_{\omega_1}$ then we will use $Q[I_{X,x}]$ otherwise just use the trivial poset. By the lemmas in this section it follows that this iteration is weakly bounding, hence b will be ω_1. In the extension assume that X is a compact space of countable tightness and weight ω_1. Let x be a limit point of the countable set $\omega \subset X$. We may assume that ω is dense in X. The diamond sequence will have chosen a space X' with the same base as X and the point x at some stage in the iteration. If we did not use $Q[I_{X,x}]$ it was because the reflected copy of X did not have countable tightness. However, uncountable tightness in compact spaces is upwards absolute (if cardinals are preserved), hence we were able to force with $Q[I_{X,x}]$. This forcing introduces a subset of ω which converges to x. The fact that $a = \omega_2$ follows from the fact that the one-point compactification of a Ψ-space is sequential but not Frechét-Urysohn. Finally, $s = \omega_2$ since we forced with Q cofinally. ❑

QUESTION 2: Does Q_{ω_1} preserve countable tightness in compact spaces? It is easy to see that C_{ω_1} does but it is certainly known that there are proper posets which do not [6].

QUESTION 3: Are there any totally mad families in this model? What if we do not use Cohen reals at ω_1 limits?

The only thing blocking a proof, for us, of an affirmative answer to the first part of the previous question is that we need a "no" answer to the following question. However it probably has a "yes" answer, but then we would ask about towers which are maximal chains in $\bigcup_{\xi < \omega_1} \mathfrak{M}_\xi$ from Proposition 2.6.

QUESTION 4: Can Q fill maximal towers?

Recall that the proper forcing axiom, PFA, implies that all compact spaces of countable tightness are sequential. However, at present the most complicated (in terms of sequential order) compact sequential space that we know under this axiom has sequential order only three!

QUESTION 4: Does PFA imply that all compact spaces of countable tightness have sequential order at most three?

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Borel Groups of Low Descriptive Complexity

FONS VAN ENGELEN

Erasmus Universiteit Rotterdam
Econometrisch Instituut, Postbus 1738
3000 DR Rotterdam, the Netherlands

ABSTRACT: Let X be a zero-dimensional first category absolute Borel set of ambiguous class two. Then X can be given the structure of a topological group if and only if X is homogeneous and $X \approx X \times X$. Thus, within the ambiguous class two, zero-dimensional groups only occur at the low levels, and the indecomposable levels of the hierarchy of difference classes $D_\alpha\Sigma_2^0$ in 2^ω.

1. INTRODUCTION

All spaces are separable and metrizable.

This paper deals with the topological structure of zero-dimensional topological groups. In [3], we gave characterizations of all zero-dimensional homogeneous absolute Borel sets. An obvious question is for which of these spaces can their homogeneity be derived from a topological group structure. This can clearly be done for those homogeneous spaces that are in the difference class $D_2(\Sigma_2^0)$, in other words, that are (at most) the intersection of a σ-compact and a complete subspace of 2^ω: the finite spaces, ω, 2^ω, $\omega \times 2^\omega$, ω^ω, $\mathbb{Q}$, $\mathbb{Q} \times 2^\omega$, and $\mathbb{Q} \times \omega^\omega$. Other zero-dimensional absolute Borel groups must be first category, since a Baire Borel group is necessarily complete. In [4] it was shown that a first category zero-dimensional absolute Borel set X has a topological group structure provided $X \times X \approx X$. In fact, by [5], for such X we have $X \times X \approx X$ if and only if X can be embedded in $2^\omega \approx \mathcal{P}(\omega)$ as an ideal on ω.

In this paper, we will show that the criterion $X \times X \approx X$ is already necessary for X to admit the structure of a topological group, if $X \in \Delta_3^0$; the general case remains open. It thus follows from the results of [5] that a zero-dimensional absolute Borel set of ambiguous class two admits a group structure if and only if X is one of the spaces of low descriptive complexity mentioned above, or X is the unique zero-dimensional homogeneous space which generates the difference class $D_\alpha(\Sigma_2^0)$ for some indecomposable ordinal $\omega \le \alpha < \omega_1$.

2. PRELIMINARIES

For standard notions from topology and descriptive set theory, see Kuratowski [8] and Kechris [7]. A subset of a space X is *clopen* if it is both closed and open in X. A space X is *homogeneous* if for each $x, y \in X$, there exists $h : X \approx X$ such

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that $h(x) = y$. Since we have restricted ourselves to metrizable spaces, we mean by a *complete* space a topologically complete metrizable space, that is, a completely metrizable space.

If $A, B \subseteq 2^\omega$ then A is *Wadge-reducible to* B (notation $A \leq_w B$) if there exists a continuous $f\colon 2^\omega \to 2^\omega$ such that $A = f^{-1}[B]$; if both $A \leq_w B$ and $B \leq_w A$ then A and B are *Wadge-equivalent*, notation $A \equiv_w B$. Let $\Gamma \subseteq \mathcal{P}(2^\omega)$. Γ is called a *Wadge class* if for some $A \subseteq 2^\omega$ (which is said to *generate* Γ), $\Gamma = \{B : B \leq_w A\}$. The *dual class of* Γ is $\check{\Gamma} = \{A : 2^\omega - A \in \Gamma\}$, and $\Delta(\Gamma)$ denotes $\Gamma \cap \check{\Gamma}$; Γ is *non-self-dual* if $\Gamma \neq \check{\Gamma}$. Γ is *continuously closed* if, whenever $A \in \Gamma$ and $B \leq_w A$, then $B \in \Gamma$. It follows from the so-called *Wadge Lemma* that if Γ is a non-self-dual and continuously closed class of Borel sets, then Γ is a Wadge class which is generated by any $A \in \Gamma - \check{\Gamma}$. The set of pairs of Borel Wadge classes $\{\Gamma, \check{\Gamma}\}$ is well-ordered by inclusion; the resulting *Wadge hierarchy* is a refinement of the usual Borel hierarchy. In fact, the additive and multiplicative Borel classes Σ^0_ξ and Π^0_ξ are themselves Wadge classes, so the refinement occurs in the ambiguous classes Δ^0_ξ. The first level of refinement consists of the so-called *small Borel classes* or *difference classes* $D_\alpha(\Sigma^0_\xi)$, defined as follows (see also Louveau[9]): for each $\alpha < \omega_1$ and each increasing sequence of Σ^0_ξ-sets $\{A_\beta : \beta < \alpha\}$, define

$$D_\alpha(A_\beta, \beta < \alpha) = \begin{cases} \bigcup_{\substack{\beta < \alpha \\ \beta \text{ even}}} (A_\beta - \bigcup_{\gamma < \beta} A_\gamma) & \text{if } \alpha \text{ is odd,} \\ \bigcup_{\substack{\beta < \alpha \\ \beta \text{ odd}}} (A_\beta - \bigcup_{\gamma < \beta} A_\gamma) & \text{if } \alpha \text{ is even;} \end{cases}$$

now $D_\alpha(\Sigma^0_\xi)$ is the class of all $D_\alpha(A_\beta, \beta < \alpha)$. We will be interested in the situation where $\xi = 2$, so the A_β are increasing sequences of σ-compacta, and the classes $D_\alpha(\Sigma^0_2)$ stratify the ambiguous Borel class Δ^0_3. The internal characterizations from [2] and [3] show that the classes $D_\alpha(\Sigma^0_2)$ and their duals are closed under homeomorphism.

Some more notation: for any class Γ, we denote by $s\sigma - \Gamma$ the class of strong countable unions from Γ, i.e., $s\sigma - \Gamma = \{X : X = \bigcup_i A_i$ for some sequence $A_i \in \Gamma$ of closed subsets of $X\}$. We say that Γ is *strongly σ-additive* if $\Gamma = s\sigma - \Gamma$. Also, Γ is *closed hereditary* if $A \in \Gamma$ whenever $X \in \Gamma$ and A is closed in X.

The following summarizes some of the results from [2] or [3] in a form that is suitable for the applications contained in this paper.

THEOREM 2.1: Let $X \subseteq 2^\omega$ be homogeneous, not locally compact, and $X \in \Delta(D_\omega(\Sigma^0_2))$. Let $\Gamma = [X]$.

(a) $\Gamma \in \{D_n(\Sigma^0_2), \check{D}_n(\Sigma^0_2)\}$ for some $1 \leq n < \omega$.

(b) X is everywhere properly Γ, i.e., $U \in \Gamma - \check{\Gamma}$(equivalently, $[U] = \Gamma$) for each nonempty clopen U in X.

(c) If $\Gamma = D_n(\Sigma^0_2)$, then $X \approx \mathbb{Q} \times Y$ for some homogeneous Y such that $[Y] = \check{D}_{n-1}(\Sigma^0_2)$ (where $\check{D}_0(\Sigma^0_2)$ is the class of all compacta); in particular, X is first category.

(d) If $\Gamma = \check{D}_n(\Sigma^0_2)$ then $X = \bigcup_i Y_i \cup P$ where $P \approx \omega^\omega$, each Y_i is closed in X, and for some homogeneous Y such that $[Y] = D_{n-1}(\Sigma^0_2)$ (where $D_0(\Sigma^0_2) = \{\emptyset\}$), $Y_i \approx Y$ for each i; in particular, X is Baire.

We will also use the following propositions (again, see [2] or [3]) which give some information on the structure of arbitrary elements of the difference classes, and some more closure properties (besides closure under homeomorphism).

PROPOSITION 2.2: Let $2 \leq n < \omega$.

(a) $D_n(\Sigma_2^0) = s\sigma - \check{D}_{n-1}(\Sigma_2^0)$.

(b) For each X, $X \in \check{D}_n(\Sigma_2^0)$ if and only if $X = Y \cup P$ for some $Y \in D_{n-1}(\Sigma_2^0)$ and some complete P.

PROPOSITION 2.3: Let $1 \leq \alpha < \omega$.

(a) If α is even then $D_\alpha(\Sigma_2^0)$ and $\check{D}_{\alpha+1}(\Sigma_2^0)$ are closed under intersection with Π_2^0-sets.

(b) $D_\alpha(\Sigma_2^0)$ is closed hereditary and strongly σ-additive.

3. THE MAIN RESULT

We must show that if $X \in \Delta_3^0$ is homogeneous zero-dimensional first category, and $X \not\approx X \times X$, then X does not admit the structure of a topological group. We will argue by contradiction, using the group structure to move "large" parts of X around. As it turns out, we can generally only control the position of countable subsets (sometimes of certain σ-compact subsets). Thus, in the spaces under consideration we want to find countable (or σ-compact) parts that are large in a certain descriptive sense.

We put $\Delta = \Delta(D_\omega(\Sigma_2^0))$. We first consider the case where $X \notin \Delta$.

LEMMA 3.1: Let $X \in \Delta_3^0 - \Delta$ be zero-dimensional and homogeneous. Then X contains a countable subset D such that for every relative Π_2^0-set G of X, if $G \supseteq D$ then $G \equiv_w X$.

Proof: Let $\Gamma = [X]$. By [3], Γ is reasonably closed (see Steel [10]), and X is everywhere properly Γ and either first category or Baire. The result now follows from [6], Lemma 2.3. ❑

If $X \in \Delta$, then by Theorem 2.1, $[X] \in \{D_n(\Sigma_2^0), \check{D}_n(\Sigma_2^0)\}$ for some $n \geq 1$ (or X is locally compact). Thus, the following lemma covers the remaining cases.

LEMMA 3.2: Let X be zero-dimensional and homogeneous, and let $\Gamma = [X]$.

(a) If $\Gamma \in \{\check{D}_{2n-1}(\Sigma_2^0), D_{2n}(\Sigma_2^0)\}$ for some $n \geq 1$, then X contains a countable subset D such that for every relative Π_2^0-set G of X, if $G \supseteq D$ then G is everywhere properly Γ.

(b) If $\Gamma \in \{D_{2n-1}(\Sigma_2^0), \check{D}_{2n}(\Sigma_2^0)\}$ for some $n \geq 1$, then X contains a σ-compact subset D such that for every relative Π_2^0-set G of X, if $G \supseteq D$ then G is everywhere properly Γ.

Proof: The proof is by induction, and relies on the facts about Δ from Section 2. If $\Gamma \in \{D_1(\Sigma_2^0), \check{D}_1(\Sigma_2^0)\}$, then X is one of $\mathbb{Q}$, $\mathbb{Q} \times 2^\omega$, or ω^ω. If $X = \omega^\omega$ we take for D any countable dense subset of X, otherwise we take $D = X$. So now assume the lemma has been proved for $\Gamma \in \{D_m(\Sigma_2^0), \check{D}_m(\Sigma_2^0)\}$, $m < n$, and $n \geq 2$.

If $\Gamma = D_n(\Sigma_2^0)$, then by Theorem 2.1 we can assume $X = \mathbb{Q} \times Y$, where Y is homogeneous, $[Y] = \check{D}_{n-1}(\Sigma_2^0)$. By the inductive hypothesis, there exists a countable or σ-compact D' which works for Y; clearly, we can assume that D' is dense in Y. If n is odd, then by the definition of $D_n(\Sigma_2^0)$ we can write $X = A \cup B$ with A being σ-compact and $B \in D_{n-1}(\Sigma_2^0)$. Now define $D = \mathbb{Q} \times D'$ if n is even, $D = (\mathbb{Q} \times D') \cup A$ if n is odd. Suppose that G is a relative Π_2^0-set in X containing D. If n is even then by Proposition 2.3 Γ is closed under intersections with Π_2^0-sets, so $G \in \Gamma$; and if n is odd then $G \cap B \in D_{n-1}(\Sigma_2^0)$ since $n-1$ is even, and it is easy to show that $G = A \cup (G \cap B) \in \Gamma$. Furthermore, $G = \bigcup_{q \in \mathbb{Q}}(\{q\} \times Y) \cap G$ is first category since $(\{q\} \times Y) \cap G$ contains $\{q\} \times D'$ and D' is dense in Y. Suppose $G \in \check{\Gamma}$, then by Proposition 2.2 we can write $G = H \cup P$ with $H \in D_{n-1}(\Sigma_2^0)$ and P complete, and since G is first category P is not dense in G. Thus, there exists a nonempty clopen subset U of G with $U \subseteq H$, and clearly some $U \cap (\{q\} \times Y) = V$ is nonempty. Then $V \in D_{n-1}(\Sigma_2^0)$. On the other hand, $G \cap (\{q\} \times Y)$ is a relative Π_2^0-set in $\{q\} \times Y$ which contains $\{q\} \times D'$, so it is everywhere properly $\check{D}_{n-1}$, and we have a contradiction. We conclude that $G \in \Gamma - \check{\Gamma}$, and the same proof shows that in fact G is everywhere properly Γ.

Finally, assume that $\Gamma = \check{D}_n(\Sigma_2^0)$. By Theorem 2.1, we can write $X = \bigcup_i Y_i \cup P$, where $P \approx \omega^\omega$, each Y_i is closed in X and homeomorphic to the same homogeneous Y, $[Y] = D_{n-1}(\Sigma_2^0)$. By the inductive hypothesis, there is a dense $D_i \subseteq Y_i$ which works for Y_i. If n is even, then by the definition of $D_n(\Sigma_2^0)$ we can write $X = A \cup B \cup C$ with A being σ-compact, $B \in D_{n-2}(\Sigma_2^0)$ (respectively, $B = \varnothing$ if $n = 2$), and C is complete. Now define $D = \bigcup_i D_i$ if n is odd, $D = \bigcup_i D_i \cup A$ if n is even. Suppose that G is a relative Π_2^0-set in X containing D. If n is odd then Γ is closed under intersections with Π_2^0-sets by Proposition 2.3, so $G \in \Gamma$; and if n is even then, as above, $G \cap (A \cup B) = A \cup (G \cap B) \in D_{n-1}(\Sigma_2^0)$, and $G \cap C$ is complete, so $G = (G \cap (A \cup B)) \cup (G \cap C) \in \Gamma$. Furthermore, G is Baire; indeed, since X is everywhere properly Γ, both $\bigcup_i Y_i$ and P are dense in X, so G is dense since it contains $\bigcup_i D_i$ whence $G \cap P$ is a dense complete subset of Y. Suppose $G \in \check{\Gamma}$, then by Proposition 2.2 we can write $G = \bigcup_i G_i$ with each $G_i \in \check{D}_{n-1}(\Sigma_2^0)$ closed in G, and since G is Baire some G_j is not nowhere dense in G. Thus, there exists a nonempty clopen subset U of G with $U \in \check{D}_{n-1}(\Sigma_2^0)$. Again using the fact that $\bigcup_i Y_i$ is dense in X, we have that some $U \cap Y_i = V$ is nonempty. Since Y_i is closed in X, $V \in \check{D}_{n-1}(\Sigma_2^0)$. On the other hand, $G \cap Y_i$ is a relative Π_2^0-set in Y_i which contains D_i, so it is everywhere properly $D_{n-1}(\Sigma_2^0)$, and we have a contradiction. Thus, $G \in \Gamma - \check{\Gamma}$, and again the same proof yields that G is everywhere properly Γ. ❑

The following lemma shows how we want to use the group structure to move the countable or σ-compact part found above around the space.

LEMMA 3.3: Let $X \subseteq 2^\omega$ admit the structure of a topological group.

(a) Suppose that X is not σ-compact, and $D \subseteq X$ is σ-compact. Then there exists a homeomorphism $h: X \to X$ such that $h[D] \cap D = \varnothing$.

(b) Let $\Gamma \subseteq \mathcal{P}(2^\omega)$ be strongly σ-additive, closed hereditary, and closed under homeomorphism. Suppose that $X \notin \Gamma$, $D \subseteq X$ is countable, and $F \in \Gamma$ is a relative Σ_2^0-set in X. Then there exists a homeomorphism $h: X \to X$ such that $h[D] \cap F = \varnothing$.

Proof: (a) Since the subgroup $\langle D\rangle$ generated by D is σ-compact, we can pick $p \in X - \langle D\rangle$. Then $pD \cap D = \emptyset$ so $h: X \approx X$, $x \mapsto px$ is as required.

(b) Write $F = \bigcup_i F_i$ with F_i closed in X; then $F_i \in \Gamma$ since Γ is closed-hereditary. Since Γ is strongly σ-additive, $Y = \bigcup_i \bigcup_{d \in D} F_i d^{-1} \in \Gamma$ so $Y \subsetneq X$, say $p \in X - Y$. Then $pD \cap F = \emptyset$ so we define h as in (a). ❑

We are now in a position to state and prove our main theorem.

THEOREM 3.4: Suppose X is a zero-dimensional absolute Borel set of the first category. If $X \in \Delta^0_3$, then X admits the structure of a topological group if and only if X is homogeneous and $X \approx X \times X$.

Proof: By [4] the condition $X \approx X \times X$ is sufficient, and by the remarks in the introduction the theorem certainly holds if $X \in D_2(\Sigma^0_2)$, so assume $X \notin D_2(\Sigma^0_2)$ is a group but $X \not\approx X \times X$. By the results of [3], $[X] = D_\alpha(\Sigma^0_2)$ for some $2 < \alpha < \omega$, and by [5] α is decomposable. Write $X = D_\alpha(A_\gamma, \gamma < \alpha)$, and let D be the countable or σ-compact subset of X given by Lemmas 3.1 and 3.2.

Case 1: $\alpha = n$ is finite and even. Define $F = A_1 - A_0 = X \cap A_1$, and $\Gamma = D_2(\Sigma^0_2)$. Then Γ is strongly σ-additive, closed hereditary, and closed under homeomorphism, $X \notin \Gamma$ since $n > 2$, and $F \in \Gamma$ is a relative Σ^0_2-set in X. Since D is countable in this case, we can apply Lemma 3.3(b) to obtain an autohomeomorphism h of X such that $h[D] \cap F = \emptyset$. Using the fact that Γ and $\check{\Gamma}$ are closed under homeomorphism, we see that $h[D]$ has the same property D has. Thus, since $X - F = D_{n-2}(A_{2+m}, m < n-2)$ is a relative Π^0_2-set in X containing $h[D]$, we have that $X - F \equiv_w X$, a clear contradiction.

Case 2: $\alpha = n$ is finite and odd. Define $F = A_0 \cup D$, then F is σ-compact, and X is not σ-compact since $n \neq 1$. Applying Lemma 3.3(a) we find $h: X \approx X$ with $h[F] \cap F = \emptyset$; again, $h[F]$ has the same property D has, because $F \supseteq D$. Thus, since $X - A_0 = D_{n-1}(A_{1+m}, m < n-1)$ is a relative Π^0_2-set in X containing $h[F]$, we have that $X - F \equiv_w X$, another contradiction.

Case 3: α is infinite. Put $\beta = \alpha$ if α is even, $\beta = \alpha + 1$ if α is odd. For each $\gamma < \beta$ define $\tilde{A}_\gamma = A_\gamma$ if α is even, $\tilde{A}_\gamma = \bigcup_{\delta < \gamma} A_\delta$ if α is odd. Then $X = D_\beta(\tilde{A}_\gamma, \gamma < \beta)$. Since α is decomposable, we can write $\alpha = \alpha_1 + \alpha_2$ with $\lim(\alpha_1)$ and $\alpha_1, \alpha_2 < \alpha$. Define $\beta_1 = \alpha_1$, and $\beta_2 = \alpha_2$ if α is even, $\beta_2 = \alpha_2 + 1$ if α is odd. Clearly, $\beta = \beta_1 + \beta_2$, $2 \leq \beta_1, \beta_2 < \alpha$, and both β_1 and β_2 are even. Now put $F = D_{\beta_1}(\tilde{A}_\gamma, \gamma < \beta_1) = X \cap \tilde{A}_{\beta_1}$, and $\Gamma = D_{\beta_1}(\Sigma^0_2)$, then Γ is strongly σ-additive, closed hereditary, and closed under homeomorphism, $X \notin \Gamma$ since $\beta_1 < \alpha$, and $F \in \Gamma$ is a relative Σ^0_2-set in X. Also, note that $X - F = D_{\beta_2}(\tilde{A}_{\beta_1 + \gamma}, \gamma < \beta_2)$ is not Wadge equivalent to X since $\beta_2 < \alpha$. Since D is countable in this case, we can apply Lemma 3.3(b) to obtain $h: X \to X$ such that $h[D] \cap F = \emptyset$. Then $X - F$ is a relative Π^0_2-set in X which contains $h[D]$, so $X - F \equiv_w X$, and we again have the required contradiction. ❑

Using [5], the following is an immediate consequence of the main theorem.

COROLLARY 3.5: Suppose $X \in \Delta^0_3$ is zero-dimensional. Then X admits the structure of a topological group if and only if X is finite, or X is one of ω, 2^ω, ω

$\times 2^\omega$, ω^ω, $\mathbb{Q}$, $\mathbb{Q}\times 2^\omega$, or $\mathbb{Q}\times\omega^\omega$, or X generates $D_\alpha(\Sigma_2^0)$ for some infinite indecomposable $\alpha < \omega_1$.

One would of course want similar results for all homogeneous zero-dimensional absolute Borel sets. We conjecture that Theorem 3.4 holds in the general case.

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Algebraic Properties of the Uniform Closure of Spaces of Continuous Functions

I. GARRIDO AND F. MONTALVO

Departamento de Matemáticas
Universidad de Extremadura
06071 Badajoz, Spain

ABSTRACT: For a completely regular space X, $C(X)$ denotes the algebra of all real-valued and continuous functions over X. This paper deals with the problem of knowing when the uniform closure of certain subsets of $C(X)$ has certain algebraic properties. In this context we give an internal condition, "property A," to characterize the linear subspaces whose uniform closure is an inverse-closed subring of $C(X)$.

For a completely regular space X, we denote by $C(X)$ the algebra of all real-valued and continuous (not necessarily bounded) functions over X. We consider the uniform topology on $C(X)$.

In the subject of *spaces of continuous functions* there are many interesting problems still unsettled. One of the most famous consists of finding an internal algebraic and topological characterization of the space $C(X)$. Many authors have treated this topic from different points of view. See, for instance, the works of Gelfand, Raikov, and Chilov [13], Fan [9], Kolhs [21], Stone [25], [26], and Arens and Kelley [4], for the compact case, and those of Anderson and Blair [3], Anderson [2], Hager [16], Mrowka [22], Henriksen [18], Henriksen and Johnson [19], Plank [23], and Császár [8], for the general case.

In the study of this problem there exist many aspects where it is very important to know algebraic properties of the uniform closure of certain subsets of $C(X)$. For instance, this becomes clear in every attempt of directly generating $C(X)$ from one of its subfamilies.

In [11] we characterized the linear subspaces of $C(X)$ whose uniform closure is closed under composition with uniformly continuous functions over $\mathbb{R}$. This was obtained by using the so-called "property C" (see Theorem 7 below), a notion we are unable to use to determine when this uniform closure has other algebraic properties such as being a subring or being inverse-closed.

Here, we are mainly interested in determining an internal condition on a subset $\mathfrak{I}$ of $C(X)$ in order to its uniform closure $\overline{\mathfrak{I}}$ be an inverse-closed subring containing all the real constant functions. (It is well known that the uniformly closed and inverse-closed subrings of $C(X)$, or more exactly the "Algebras over X" defined by Hager and Johnson in [17], play an important role in all this framework as can

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be seen, for instance, in Hager [16], Henriksen and Johnson [19], Isbell [20], Blasco [5], [6], [7], Garrido [10], and Steiner and Steiner [24]).

Clearly, a sufficient condition for our purpose will be when $\overline{\mathfrak{I}} = C(X)$, that is, when $\mathfrak{I}$ is uniformly dense in $C(X)$. In [11] we make a general study of the uniform approximation in $C(X)$ and we obtain in particular the next result about uniform density. Our notation and terminology will be as in Gillman and Jerison [14].

THEOREM 1: [11] Let $\mathfrak{I}$ be a linear subspace of $C(X)$. The following conditions are equivalent:

(a) $\mathfrak{I}$ is uniformly dense in $C(X)$.

(b) For each countable cover $\{C_n\}_{n\in\mathbb{Z}}$ of X by cozero-sets such that $C_n \cap C_m = \varnothing$ if $|n-m| > 1$ (we call 2-finite cover) there is $h \in \mathfrak{I}$ with $|h(x) - n| < 2$ when $x \in C_n$ $(n \in \mathbb{Z})$.

So, the above condition (b) over a linear subspace $\mathfrak{I}$ makes $\overline{\mathfrak{I}}$ to have the required algebraic properties. But this sufficient condition is not internal because we use the cozero-sets of all the continuous functions on X. To avoid that, we give the next definition.

DEFINITION 2: A subset $\mathfrak{I}$ of $C(X)$ has the "*property* A" if for each 2-finite cover $\{C_n\}_{n\in\mathbb{Z}}$ of X by cozero-sets of functions in $\mathfrak{I}$, there is $h \in \mathfrak{I}$ with $|h(x) - n| < 2$ when $x \in C_n$ $(n \in \mathbb{Z})$.

Now, by means of this property we shall state our main result.

THEOREM 3: Let $\mathfrak{I}$ be a linear sublattice of $C(X)$ containing all the real constant functions. The following conditions are equivalent:

(a) $\mathfrak{I}$ has the property A;

(b) $\overline{\mathfrak{I}}$ is closed under composition with continuous and monotonic functions defined on real open intervals;

(c) $\overline{\mathfrak{I}}$ is an inverse-closed subring of $C(X)$.

Proof: (a) implies (b). Let $f \in \overline{\mathfrak{I}}$ and φ a monotonic (for instance nondecreasing) function defined on the real open interval I containing $f(X)$.

Let $\varepsilon > 0$ and $C_n = \{x \in X: (n-1)\varepsilon < \varphi \circ f(x) < (n+1)\varepsilon\}$, $n \in \mathbb{Z}$. We shall prove that there exist $h \in \mathfrak{I}$ such that $|h(x) - n| < 3$ when $x \in C_n$. This will be enough to see that $\varphi \circ f$ belongs to $\overline{\mathfrak{I}}$. Indeed, if $x \in X$ then $x \in C_n$ for some $n \in \mathbb{Z}$ and then $|\varepsilon h(x) - \varphi \circ f(x)| \le |\varepsilon h(x) - n\varepsilon| + |n\varepsilon - \varphi \circ f(x)| \le 3\varepsilon + \varepsilon$. Note that from the hypothesis such a function h must exist whenever every C_n is a cozero-set of some function in $\mathfrak{I}$, but in general this can fail.

Let $C_n' = \{x \in X: \alpha_{n-1} < f(x) < \alpha_{n+1}\}$ where $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a nondecreasing sequence in $\mathbb{R} \cup \{-\infty, \infty\}$ such that:

$$\varphi(\alpha_n) = n\varepsilon \text{ if } n\varepsilon \in \varphi(I)$$

$$\alpha_n = -\infty \text{ if } n\varepsilon \notin \varphi(I) \text{ and } n\varepsilon \le \inf\varphi(I)$$

$$\alpha_n = \infty \text{ if } n\varepsilon \notin \varphi(I) \text{ and } n\varepsilon \ge \sup\varphi(I).$$

It is easy to check that $\alpha_n - \alpha_{n-1} > 0$ when α_n is a real number, $C_n \subset C_n'$ for every $n \in \mathbb{Z}$ and $\{C_n'\}_{n\in\mathbb{Z}}$ is a 2-finite cover of X.

For each $n \in \mathbb{Z}$ choose an arbitrary real number $\varepsilon_n > 0$ but such that $\varepsilon_n < (\alpha_n - \alpha_{n-1})/4$ if $(\alpha_n - \alpha_{n-1})/4$ is well defined, and $\varepsilon_n < (\alpha_{n+1} - \alpha_n)/4$ if $(\alpha_{n+1} - \alpha_n)/4$ is also well defined. Then take $g_n \in \Im$ with $|g_n - f| < \varepsilon_n$ (recall that $f \in \overline{\Im}$).

Now, let $D_n = \{x \in X\colon \alpha_{n-1} + \varepsilon_n < g_n(x) < \alpha_{n+1} - \varepsilon_n\}$, $n \in \mathbb{Z}$. Since $\Im$ is a linear sublattice containing all the real constant functions, every D_n is a cozero-set of some function in $\Im$. Moreover, for every $n \in \mathbb{Z}$, we have $D_n \subset C_n' \subset D_{n-1} \cup D_n \cup D_{n+1}$. Indeed, the first inclusion is clear and for the second one we can consider three cases:

(1) "$\alpha_{n-1} + 2\varepsilon_n < f(x) < \alpha_{n+1} - 2\varepsilon_n$." Then $\alpha_{n-1} + \varepsilon_n < g_n(x) < \alpha_{n+1} - \varepsilon_n$ and hence $x \in D_n$.

(2) "$\alpha_{n-1} < f(x) \le \alpha_{n-1} + 2\varepsilon_n$." In this case α_{n-1} must be in particular a real number and so we have $\alpha_{n-2} + \varepsilon_{n-1} < \alpha_{n-1} - \varepsilon_{n-1} < g_{n-1}(x) < f(x) + \varepsilon_{n-1} \le \alpha_{n-1} + 2\varepsilon_n + \varepsilon_{n-1} < \alpha_n - \varepsilon_{n-1}$. Thus $x \in D_{n-1}$.

(3) "$\alpha_{n+1} - 2\varepsilon_n \le f(x) < \alpha_{n+1}$." Using an analogous argument as the preceding one we can show that now $x \in D_{n+1}$.

Thus $\{D_n\}_{n\in\mathbb{Z}}$ is a 2-finite cover of X by cozero-sets of functions in $\Im$. And finally, by (a) there exists $h \in \Im$ such that $|h(x) - n| < 3$ if $x \in C_n$, as we wanted.

(b) implies (c). In order to see that $\overline{\Im}$ is a subring it is enough to prove that $f^2 \in \overline{\Im}$ whenever $f \in \overline{\Im}$. If $f > 0$, then $f^2 = \varphi \circ f$ where φ is the continuous and monotonic function $\varphi(t) = t^2$ defined over $(0, \infty)$ and therefore, f^2 belongs to $\overline{\Im}$. Since $\overline{\Im}$ is a linear sublattice containing all the real constant functions, the general case follows from the equality $f^2 = |f|^2 = (|f| + 1)^2 - 2|f| - 1$.

On the other hand, if $f \in \overline{\Im}$ and $Z(f) = \varnothing$, then the function $1/f^2$ belongs to $\overline{\Im}$ because $1/f^2 = \psi \circ f^2$ where ψ is the continuous and monotonic function $\psi(t) = 1/t$ defined over $(0, \infty)$. As a consequence, $1/f \in \overline{\Im}$ since $1/f = (1/f^2) \cdot f$ and so $\overline{\Im}$ is inverse-closed.

(c) implies (a). Let $\{C_n\}_{n\in\mathbb{Z}}$ be a countable 2-finite cover of X by cozero-sets of functions in $\Im$. We shall proceed in two steps.

First step: We define $D_0 = \bigcup_{n\le 0} C_n$ (note that D_0 is also a cozero-set of some function in $\overline{\Im}$). From the properties of $\overline{\Im}$ we can choose $g_0, g_1, \ldots, g_n, \ldots$, functions in $\overline{\Im}$ with $0 \le g_n \le 1/2^n$ such that $D_0 = \operatorname{coz}(g_0)$ and $C_n = \operatorname{coz}(g_n)$ for every $n = 1, 2, \ldots$. Since the function $g = \sum g_n$ belongs to $\overline{\Im}$ and $g(x) > 0$ for every $x \in X$, then the sequence $\{v_n\}_{n=0}^{\infty}$ defined by $v_n = g_n/g$ for each n, is a partition of unity by functions in $\overline{\Im}$ with $D_0 = \operatorname{coz}(v_0)$ and $C_n = \operatorname{coz}(v_n)$ for every $n = 1, 2, \ldots$.

If we consider $h = \sum\left(1/(n+1)^2\right)v_n$ then h is also in $\overline{\Im}$ and satisfies:

$$h(x) = 1 \text{ on } D_0 - C_0$$

$$1/2^2 \le h(x) \le 1 \text{ on } C_0$$

$$1/(n+2)^2 \le h(x) \le 1/n^2 \text{ on } C_n, \quad n \ge 1.$$

Now, we can apply the classical Weierstrass approximation theorem to derive that $h^{1/2}$ belongs to $\overline{\Im}$ because $0 \le h \le 1$. So, the function $f_1 = (1/h^{1/2}) - 1$ is in $\overline{\Im}$ and satisfies:

$$f_1(x) = 0 \quad \text{on} \quad \bigcup_{n\le 0} C_n - C_0$$

$$0 \le f_1(x) \le 1 \text{ on } C_0$$

$$n-1 \le f_1(x) \le n+1 \text{ on } C_n,\ n \ge 1.$$

Second step: By mimicry of the above step we construct a function $f_2 \in \overline{\mathfrak{I}}$ with:

$$f_2(x) = 0 \text{ on } \bigcup_{n \ge 0} C_n - C_0$$

$$-1 \le f_2(x) \le 0 \text{ on } C_0$$

$$n-1 \le f_2(x) \le n+1 \text{ on } C_n,\ n \le -1.$$

Finally the function $f_1 + f_2 \in \overline{\mathfrak{I}}$ satisfies $|(f_1 + f_2)(x) - n| < 1$ when $x \in C_n$, for every $n \in \mathbb{Z}$. And we complete the proof if we take $h \in \mathfrak{I}$ with $|(f_1 + f_2) - h| < 1$. ❑

We would like to remark that the above result can be extended to the linear subspaces of $C(X)$ that are not necessarily sublattices. In order to do that, we need to change the definition of the property A only a little (see [12]). For further applications of this result, such as the *generation of Algebras on X*, or the study of zero-sets and cozero-sets spaces defined by Gordon [15] and Alexandroff [1], respectively, or the problem of extension of continuous functions, etc., we also refer to [12].

Now we shall show how to obtain, as an easy corollary of the above theorem, the well-known result by Henriksen and Johnson [19] about the *Algebras on Lindelöf spaces.*

COROLLARY 4: (Henriksen and Johnson [19]) If X is a Lindelöf space and $\mathfrak{I}$ is an Algebra on X (that is, $\mathfrak{I}$ is a uniformly closed and inverse-closed subring of $C(X)$ containing all the real constant functions and separating points and closed sets of X), then $\mathfrak{I} = C(X)$.

Proof: First note, that under these conditions $\mathfrak{I}$ is also a sublattice of $C(X)$. And so, from Theorem 3, $\mathfrak{I}$ has the property A. Now, if we compare property A and the condition of uniform density in Theorem 1 we shall derive that $\mathfrak{I}$ is uniformly dense in $C(X)$, and therefore, $\mathfrak{I}$ will be $C(X)$ if we see that every cozero-set in X is the cozero-set of some function in $\mathfrak{I}$.

Thus, if C is a cozero-set in X then $C = \bigcup_{i \in I} C_i$ where C_i are cozero-sets of functions in $\mathfrak{I}$ because $\mathfrak{I}$ determines the topology on X. Since every cozero-set is an F_σ-set and X is a Lindelöf space then C is also Lindelöf and so, we have $C = \bigcup_{n \in \mathbb{N}} C_{i_n}$. And this completes the proof because any countable union of cozero-sets of functions in $\mathfrak{I}$ is also a cozero-set of some function in $\mathfrak{I}$. ❑

REMARK: In [23], Plank gives a different version of the above corollary that, as it is said there, has a slightly more algebraic flavor. It consists of an (internal) algebraic characterization of $C(X)$, for X a Lindelöf space, when it is considered on $C(X)$ the structure of Φ-algebra (for that notion see [19] or [23]). This characterization and Corollary 4 are in fact the same result but given in different frameworks and then, it also can be obtained as a consequence of Theorem 3. In order to check that both results are equal, it is enough to note that if $\mathfrak{I}$ is a Φ-algebra of real-valued functions, that is, if $\mathfrak{I}$ can be represented as an algebra of continuous functions on a (realcompact) space $\mathcal{R}(\mathfrak{I})$, then $\mathfrak{I}$ satisfies the conditions by Plank if and only if $\mathcal{R}(\mathfrak{I})$ is a Lindelöf space and $\mathfrak{I}$ is an Algebra on $\mathcal{R}(\mathfrak{I})$.

We are going to finish this note by setting some open questions.

QUESTION 5: Let $\mathfrak{I}$ be a linear sublattice of $C(X)$ containing all the real constant functions. Is it possible to give an internal, necessary and sufficient, condition over $\mathfrak{I}$ making its uniform closure to be a subring?

That is, we are looking for an analogous result to Theorem 3, but deleting the condition of being inverse-closed. It is clear that the property A will be sufficient for that, but it is not necessary as the next example shows.

EXAMPLE 6: [11] Let $\mathfrak{I}$ be the subset of $C(\mathbb{R})$ defined by

$$\mathfrak{I} = \left\{\sum_{i=1}^{n} f_i p_i : f_i \in C_0(\mathbb{R}) \text{ and } p_i \text{ is a polynomial, } i = 1, \ldots, n\ (n \in \mathbb{N})\right\}$$

where $C_0(\mathbb{R})$ denotes the set of all continuous functions over $\mathbb{R}$ vanishing at infinity. We saw in [11] that $\mathfrak{I}$ is a uniformly closed linear sublattice and subring containing all the real constant functions. But $\mathfrak{I}$ has not the property A because it is not inverse-closed. To show that, it is enough to check that the function $f(x) = \exp(-x^2)$ is in $\mathfrak{I}$ (in fact $f \in C_0(\mathbb{R})$) but $1/f$ is not.

On the other hand, in [11] we defined and studied another internal condition called "*property* C" that can be characterized in the following way.

THEOREM 7: [11] A linear subspace $\mathfrak{I}$ of $C(X)$ has the property C if and only if $\overline{\mathfrak{I}}$ is closed under composition with uniformly continuous functions over $\mathbb{R}$.

Also, we need to recall the next result about the property C.

PROPOSITION 8: [11] If the uniform closure of a linear subspace $\mathfrak{I}$ is a subring and sublattice containing all the real constant functions, then $\mathfrak{I}$ has the property C.

Thus, from the above Proposition 8 it follows that property C is a necessary condition for our purpose. And now, from Theorem 7 we can see that, unfortunately, this property is not sufficient even when the linear sublattice $\mathfrak{I}$ is itself a subring. For that, we shall use an example due to Isbell that, in particular, makes clear how the uniform closure of a subring need not be a subring.

EXAMPLE 9: (Isbell [20]) Let X be the subspace of the plane consisting of the horizontal lines $L_n = \{(x, n): x \in \mathbb{R}\}$ for $n = 2, 3, \ldots$, and let $\mathfrak{I}$ the set of all continuous functions f on X such that for some m, f is constant on each L_n for $n > m$. Obviously $\mathfrak{I}$ is a linear sublattice and subring containing all the real constant functions. Moreover, $\mathfrak{I}$, and hence $\overline{\mathfrak{I}}$, are closed under composition with uniformly continuous functions defined on $\mathbb{R}$ and then, from Theorem 7, $\mathfrak{I}$ has the property C. But $\overline{\mathfrak{I}}$ is not a subring. Indeed, if we take the function g mapping each L_n homeomorphically onto the real interval $(n - 1/n, n + 1/n)$, then g can be approximated by functions in $\mathfrak{I}$. But g^2 is not in $\overline{\mathfrak{I}}$, because this function maps every L_n onto the interval $(n^2 + 1/n^2 - 2, n^2 + 1/n^2 + 2)$ whose length is just 4.

All the above tells us that we are looking for a property between C and A. On the other hand, it is easy to check that this unknown condition can be externaly characterized in terms of composition with continuous functions over $\mathbb{R}$. In fact, the uniform closure of a linear subspace is a linear sublattice and subring containing all the real constant functions if and only if this uniform closure is closed under composition with the functions of the subset of $C(\mathbb{R})$ defined in Example 6 (see [11]). This subset is a wide class of continuous functions, but it is not all $C(\mathbb{R})$. Thus, the following question arises.

QUESTION 10: Let $\Im$ be a linear sublattice of $C(X)$ containing all the real constant functions. Is it possible to give an internal, necessary and sufficient, condition over $\Im$ making its uniform closure to be closed under composition with all the continuous functions over $\mathbb{R}$?

From the above study it follows at once that Questions 5 and 10 are not equivalent. The property C is again necessary but not sufficient. And the property A is only sufficient because it is easy to find uniformly closed linear sublattices that are closed under composition with all $C(\mathbb{R})$ but not inverse-closed, that is, without the property A. For instance if X is a nonpseudocompact space and $p \in \beta X - \upsilon X$ (where βX and υX denote, respectively, the Stone-Čech compactification and the Hewitt-Nachbin realcompactification of X), then the set $\Im$ of all the continuous functions over X having continuous extension to $X \cup \{p\}$, satisfies the required.

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The Intermediate Value Theorem for Polynomials over Lattice-ordered Rings of Functions

MELVIN HENRIKSEN,[a] SUZANNE LARSON,[b] AND JORGE MARTINEZ[c]

[a]*Department of Mathematics*
Harvey Mudd College
Claremont, California 91711

[b]*Department of Mathematics*
Loyola Marymount University
Los Angeles, California 90045
and
[c]*Department of Mathematics*
University of Florida
Gainesville, Florida 32611

ABSTRACT: The classical intermediate value theorem for polynomials with real coefficients is generalized to the case of polynomials with coefficients in a lattice-ordered ring that is a subdirect product of totally ordered rings. Several candidates for a generalization are investigated, and particular attention is paid to the case when the lattice-ordered ring is the algebra $C(X)$ of continuous real-valued functions on a completely regular topological space X. For all but one of these generalizations, the intermediate value theorem holds only if X is an F-space in the sense of Gillman and Jerison. Surprisingly, for the most interesting of these generalizations, if X is compact, the intermediate value theorem holds only if X is an F-space and each component of X is an hereditarily indecomposable continuum. It is not known if there is an infinite compact connected space in which this version of the intermediate value theorem holds.

1. INTRODUCTION

The classical intermediate value theorem states that if a continuous real-valued function f of a real variable has opposite signs at points u and v, then $f(w) = 0$ for some w between u and v. We have been unable to find any literature on analogues of the intermediate value theorem in case the domain is a partially ordered ring.

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[a]E-mail: Mhenriksen@hmc.edu.
[b]E-mail: Slarson@lmumail.lmu.edu.
[c]E-mail: Martinez@math.ufl.edu.

To reduce generality to a manageable degree, we confine ourselves to the case when the domain is an f-ring (i.e., a lattice-ordered ring which is a subdirect product of totally ordered rings), and the "continuous" functions are polynomials. Even then, there is some doubt about what should be meant by THE intermediate value theorem, and three versions are considered below. What we regard as the most natural one is presented first. Our main results will seem counter-intuitive at first because we usually associate the intermediate value theorem with connectedness. It turns out that the connectedness of the range space is what is involved; not that of the domain.

Suppose A is a commutative f-ring with identity element, let $A[t]$ denote the ring of polynomials with coefficients in A, and let G denote a subset of $A[t]$ closed under translation by elements of A; that is, if $p(t) \in G$ and $a \in A$, then $p(t-a) \in G$. Note that for each positive integer n, the set of polynomials of degree no larger than n and the set of monic polynomials of degree n are closed under translation.

(1) If, for every $p(t) \in G$, and distinct elements $u,v \in A$ such that $p(u) > 0$ and $p(u) < 0$, there is a $w \in A$ such that $p(w) = 0$ and $u \wedge v \leq w \leq u \vee v$, we say the *intermediate value theorem (IVT) holds for G*. In case $G = A[t]$, then A is called an *IVT ring*.

(2) If the ring $C(X)$ of all continuous real-valued functions on a (Tychonoff) space X is an IVT ring, then we call X an *IVT space*.

In the latter case, $p(u) > 0$ means that $p(u(x)) \geq 0$ for all $x \in X$, and $p(u(x)) > 0$ for some $x \in X$. The meaning of $p(v) < 0$ is similar, and writing $p(w) = 0$ means $p(w(x)) = 0$ for all $x \in X$.

In this paper we consider f-rings in which the intermediate value theorem holds for certain $G \subseteq A[t]$, and in particular, we consider IVT rings and spaces. In the second section we give basic properties of such and use these to give examples of several types of IVT rings and spaces. We show that the intermediate value theorem holds for linear polynomials in an f-ring A if and only if A is 1-convex. (That is, a is a multiple of b whenever $0 \leq a \leq b$.) For a (Tychonoff) space, this is equivalent to saying that in $C(X)$, the intermediate value theorem holds for linear polynomials if and only if X is an F-space (in the sense used in [7]). We show that every compact zero-dimensional F-space is an IVT space. Also, we show that every C-embedded subspace of an IVT space is an IVT space, and every Dedekind-MacNeille complete f-algebra is an IVT ring.

It is an open question as to whether or not there is an infinite connected IVT space. In the third section we show that if there is a connected IVT space, then it must be hereditarily indecomposable. In fact, every continuum contained in an IVT space is hereditarily indecomposable. We also give examples of connected F-spaces that are not IVT spaces, and of hereditarily indecomposable continua that are F-spaces. The fourth section gives two variations on the concept of IVT rings and spaces and their properties.

Preliminaries

A ring A is a *subdirect product* of the family of rings $\{A_x: x \in X\}$ for some index set X, if A is a subring of the direct product $\prod_{x \in X} A_x$ such that the projection of A onto each A_x is a surjection. An *f-ring* is a lattice-ordered ring which is a subdirect product of totally ordered rings. An *f-algebra* is an f-ring which is also an algebra over the real field $\mathcal{R}$, in which a positive scalar multiple of a positive element is positive. For general information on f-rings and f-algebras, see [2]. Given an f-ring A, we let $A^+ = \{a \in A: a \geq 0\}$, and for an element $a \in A$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. A ring ideal I of an f-ring A is an *ℓ-ideal* if $|a| \leq |b|$, $b \in I$ implies $a \in I$ or equivalently, if I is the kernel of some lattice-preserving ring homomorphism (ℓ-homomorphism).

Suppose A is an f-ring and I is an ideal of A. The ideal I is *semiprime* (respectively, *prime*) if $J^2 \subseteq I$ (respectively, $JK \subseteq I$) implies $J \subseteq I$ (respectively, $J \subseteq I$ or $K \subseteq I$) for ideals J,K and the f-ring A is called semiprime (respectively, prime) if $\{0\}$ is semiprime (respectively, prime). It is well known that in an f-ring, an ℓ-ideal I is semiprime if and only if it is an intersection of the prime ℓ-ideals which are minimal with respect to containing I. If P is a prime ℓ-ideal of the f-ring A, then A/P is a totally ordered prime ring and all ℓ-ideals of A containing P form a chain (see [2, Chap. 8]). On several ocassions, we will make use of the following well-known result.

A semiprime f-ring is a subdirect product of totally ordered prime rings. (1.1)

The f-ring A is called *archimedean* if for any $0 < a \in A+$, the set $\{na: n \in \mathbf{N}\}$ is not bounded above. (Here and elsewhere, $\mathbf{N}$ denotes the set of positive integers.) A commutative f-ring A is called *1-convex* if $0 \leq a \leq b$ implies that there is a $w \in A$ such that $a = wb$, or equivalently if every ring ideal of A is an ℓ-ideal. In an f-ring with identity element, the w of the definition may be chosen so that $0 \leq w \leq 1$. Further equivalent properties can be found in [6], [14], and [16]. A known fact about 1-convex f-rings that we will make use of is: .

For any element a of a 1-convex f-ring with identity element, there is another element c, such that $a = c|a|$ and $|a| = ca$ and $|c| \leq 1$. (1.2)

Now suppose X is a topological space. Let $C(X)$ denote the f-ring of all real valued continuous functions defined on X, under pointwise operations. A subset Y of X is a *zeroset* if $Y = \{x \in X: f(x) = 0$ for some $f \in C(X)\}$. A *cozeroset* is the complement of a zeroset. Given an $f \in C(X)$, we let $Z(f)$ (respectively, $\operatorname{coz}(f)$) denote the zeroset (respectively, cozeroset) determined by f. A space X is Tychonoff if it has a base of cozerosets. Following standard notation, we let βX denote the *Stone-Čech* compactification of X. For background information on $C(X)$ and its Stone-Čech compactification, see [7, Chapter 6].

A subspace Y of X is *C-embedded* (respectively, *C^*-embedded*) in X if every $f \in C(Y)$ (respectively, bounded $f \in C(Y)$) can be extended to a function in $C(X)$. If every finitely generated ideal of $C(X)$ is principal, we call X an *F-space*. A number of conditions are known to be equivalent to X being an F-space (see [7, Chapter 14]). In particular, we note that the following are equivalent:

(a) X is an F-space;
(b) Every cozeroset of X is C^*-embedded;

(c) $C(X)$ is 1-convex.

In this article, all f-rings considered are commutative with identity element and all topological spaces that arise are Tychonoff.

2. IVT RINGS

In this section, we present some basic properties of IVT rings and spaces and use these properties to give examples of IVT rings and spaces. In particular, we show that a compact zero-dimensional F-space is an IVT space.

For any f-ring A with identity element, we let $A^* = \{a \in A: |a| \le n \cdot 1$ for some $n \in \mathbf{N}\}$, and call A^* the *subring of bounded elements* of A. We begin by taking note of the following. The proof is straightforward and has been omitted.

PROPOSITION 2.1: Let A be an *f-ring*.

(a) If A is an IVT ring and B is a convex f-subring of A (i.e., B is an f-subring such that $0 \le a \le b$ with $a \in A$, $b \in B$ implies $a \in B$) then B is also an IVT ring.

(b) If A is an IVT ring, then so is A^*.

(c) A direct product of IVT rings is an IVT ring.

The converse of part (b) of the proposition does not hold, as we will see after the next theorem. However, because $C^*(X)$ and $C(\beta X)$ are isomorphic, it follows from the proposition that if X is an IVT space, then so is its Stone-Čech compactification βX. Whether the converse holds is an open question.

We now characterize commutative semiprime f-rings with identity element for which the intermediate value theorem holds for linear polynomials.

THEOREM 2.2: If A is a commutative semiprime f-ring with identity element, then the IVT for linear polynomials holds if and only if A is 1-convex.

Proof: Suppose first that the IVT for linear polynomials holds in A, and assume $0 < a < b$ in A. Let $p(t) = a - bt$. Because $p(0) = a > 0$, and $p(1) = a - b < 0$, there is a $w \in A$ such that $0 \le w \le 1$ and $p(w) = a - bw = 0$. Thus, $a = bw$ and A is 1-convex.

Suppose, conversely that A is 1-convex and that $p(t) = a + bt$ is a linear polynomial in $A[t]$ such that for distinct u,v in A, $p(u) > 0$ and $p(v) < 0$. Then, $a < -bv \le |bv|$ and hence $a^+ \le |bv|$. (To see that the latter holds, note that $a = a^+ - a^-$, and in any totally ordered ring, $a = a^+$ or $a^+ = 0$.) A similar argument shows that $a^- \le |bu|$. Because A is 1-convex, there are $w,w' \in A^+$ such that $a^+ = w|bv| = w|b||v|$ and $a^- = w'|b||u|$. So, $a = a^+ - a^- = (w|v| - w'|u|)|b|$. Using 1.2 yields an r such that $|b| = rb$. Letting $z = -(w|v| - w'|u|)r$, we see that $p(z) = 0$.

Now consider $z' = (z \wedge (u \vee v)) \vee (u \wedge v)$. Clearly, $u \wedge v \le z' \le u \vee v$. We will show that $p(z') = 0$. By 1.1, A is the subdirect product of totally ordered prime rings A_x for some index set X. For any $a \in A$, we let a_x denote the projection of a onto A_x. To show that $p(z') = 0$, it suffices to show that if $p(z')_x = 0$ for each x. If $b_x > 0$, then since $a_x + b_x v_x \le 0 = a_x + b_x z_x \le a_x + b_x u_x$ and A_x is prime, it follows that $v_x \le z_x \le u_x$. Next, note that if $b_x < 0$, a similar argument shows that $u_x \le z_x \le v_x$. So if either $b_x > 0$ or $b_x < 0$, then $z_x' = z_x$ and hence $p(z')_x = 0$. Otherwise, $b_x =$

0 and $a_x + b_x z_x = 0$ implies $a_x = 0$, whence $p(z')_x = a_x + b_x z_x' = 0$. It follows that the *IVT* for linear polynomials holds in A. ❑

The converse of part (*b*) of Proposition 2.1 need not hold because a totally ordered ring may fail to be 1-convex even if its subring of bounded elements is 1-convex. For example, if A is the ring $\mathcal{R}[x]$ of polynomials lexicographically ordered with $1 \ll x \ll x^2 \ll \cdots \ll x^n \ll \cdots$, then A fails to be 1-convex, while $A^* = \mathcal{R}$. There is also an archimedean semiprime f-ring A that fails to be an IVT ring while A^* is an IVT ring. Consider $B = \{f \in C(\omega) : f$ coincides eventually with a polynomial$\}$. Because B fails to be 1-convex, it is not an IVT ring, while $B^* = \{f \in C(\omega) : f$ is eventually constant$\}$ is an IVT ring. (Here and elsewhere, ω denotes the discrete space of finite ordinals.)

An f-ring A is called *square-root closed* if for any $a \in A^+$, there is a $b \in$ A such that $b^2 = a$. If the intermediate value theorem holds for certain polynomials, then (square) roots must exist.

THEOREM 2.3: Suppose A is a commutative semiprime f-ring with identity element.

(a) If in A, the IVT for monic polynomials of degree n holds, then for any $s \in A^+$, there is an $r \in A$ such that $r^n = s$.

(b) If A is an f-algebra over the field of rational numbers, then the IVT for monic quadratic polynomials holds if and only if it is square-root closed.

Proof: (a) We may assume $s > 0$. Apply the IVT with $p(t) = t^n - s$, $u = 1 + s$, and $v = 0$.

(b) Part (a) shows that if in A the IVT for monic quadratic polynomials holds, then it is square-root closed.

Now suppose A is square-root closed, $p(t) \in A[t]$, and u,v belong to A with $p(u) > 0$, $p(v) < 0$. By completing the square, we may assume $p(t)$ is of the form $p(t) = t^2 - f$ for some $f \in A$. By 1.1, A is a subdirect product of totally ordered prime rings A_x for some index set X. For any $a \in A$, we let a_x denote the projection of a onto A_x. Note that for each x, $-f_x \leq v_x^2 - f_x \leq 0$. So $f \geq 0$, and there is a $b \in A^+$ such that $f = b^2$. Thus $p(t) = t^2 - b^2$. Since $p(u) > 0$ and $p(v) < 0$, $v^2 < b^2 < u^2$. Let $w = (b \wedge u^+) - (b \wedge u^-)$. We will show that $p(w) = 0$ and $u \wedge v \leq w \leq u \vee v$. Note first that for any x, $u_x = u_x^+$ and $u_x^- = 0$, or, $u_x = -u_x^-$ and $u_x^+ = 0$. It follows that for each x,

$$w_x = \begin{cases} b_x & \text{if } u_x > 0 \\ -b_x & \text{if } u_x < 0 \\ 0 & \text{if } u_x = 0, \end{cases}$$

and $w^2 = b^2$, so $p(w) = 0$. It is easy to check that for each x, $u_x \wedge v_x \leq w_x \leq u_x \vee v_x$. Therefore, $u \wedge v \leq w \leq u \vee v$ in A. Thus in A, the IVT for monic quadratic polynomials holds. ❑

Before giving our next result, we need to recall a known fact about 1-convex f-rings: every ℓ-homomorphic image of a 1-convex f-ring is 1-convex [14, 2.3].

THEOREM 2.4: Every semiprime ℓ-homomorphic image of a semiprime IVT ring is an IVT ring.

Proof: Suppose $B = A/I$ is a semiprime ℓ-homomorphic image of the semiprime IVT ring A. Suppose $p(t) = \sum_{i=0}^{n} a_i t^i \in A[t]$, $p(u + I) > 0$, and $p(v + I) < 0$ mod I for some $u,v \in A$. It follows from 2.2 and the fact that ℓ-homomorphic images of 1-convex f-rings are 1-convex, that B is 1-convex. So by 1.2 there is a $c \in A$ such that $|v - u| = c(v - u)$. Let $q(t) \in A[t]$ be defined by $q(t) = |v - u|p(t) + c(v - t)(p(v) - p(u))^+$. Now $q(v) = |v - u|p(v)$ and $q(u) = |v - u|p(u) + c(v - u)(p(v) - p(u))^+ = |v - u|[p(u) + (p(v) - p(u))^+] = |v - u|[p(u) \vee p(v)]$. Hence:

$$q(v) \leq q(u). \tag{*}$$

Let $k = q(v)^+ \wedge q(u)$, and observe, using (*), that $q(u) - k \geq 0$ and $q(v) - k \leq 0$. Since A is an IVT ring, there is a $w \in A$ such that $q(w) = k$ and $u \wedge v \leq w \leq u \vee v$.

Because $q(v) = |v - u|p(v)$ and $p(v) < 0$ mod I, we must have $q(v)^+ \in I^+$. Because $q(u) = |v - u|[p(v) \vee p(u)]$ and $p(u) > 0$ mod I, we must have $q(u) \geq 0$ mod I. It follows that $k \in I$. So $q(w) = 0$ mod I. Since $(p(v) - p(u))^+ = 0$ mod I, we have $q(w) = |v - u|p(w) = 0$ mod I. By 1.1, A/I is a subdirect product of totally ordered prime rings $\{A_x\}_{x \in X}$ for some index set X. For $b \in A/I$, let b_x denote the projection of b in A_x. If $v_x \neq u_x$, then $p(w)_x = 0$. If $v_x = u_x$, then because $u \wedge v \leq w \leq u \vee v$, $w_x = u_x = v_x$. Since $p(u) > 0$ mod I and $p(v) < 0$ mod I, it follows that $p(w)_x = p(u)_x = p(v)_x = 0$ mod I. Since $p(w)_x = 0$ for each x, $p(w) = 0$ in A/I. ❑

A consequence of this theorem is given next. Part (a) follows directly from the theorem and part (b) follows from Proposition 2.1(b) and the well-known fact that every cozeroset of an F-space is C^*-embedded.

COROLLARY 2.5: Let X be an IVT.

(a) Every C-embedded subspace of X is an IVT space. In particular, if X is compact, then every closed subspace of X is an IVT.

(b) For every cozeroset U of X, $C^*(U)$ is an IVT ring.

Recall that a topological space X is *zero-dimensional* if it has a base of clopen sets and is *strongly zero-dimensional* if βX is zero-dimensional. Also, a *regular* element of a commutative f-ring is one which is not a divisor of zero. The following theorem will be useful later.

THEOREM 2.6: Suppose X is a compact zero-dimensional F-space, $A = C(X)$, and the IVT for monic polynomials holds in A. If $p(t) = \sum_{i=0}^{n} a_i t^i \in A[t]$ and $u,v \in A$ are such that $p(u) > 0$, $p(v) < 0$ then there is a $w \in A$ such that $p(w) = 0$ on $\mathrm{coz}(a_n)$ and $u \wedge v \leq w \leq u \vee v$. In particular, if the IVT for monic polynomials holds, then it also holds for all polynomials with a regular leading coefficient.

Proof: Because X is a zero-dimensional F-space, there is a unit $h \in A$ such that $|a_n| = ha_n$. So we may assume that $a_n > 0$. Since zerosets of a zero-dimensional space can be separated by clopen sets, we may choose for each $k \in \mathbf{N}$, a clopen set V_k such that $\{x : a_n(x) \geq 1/2^k\} \subseteq V_k$ and $V_k \cap \{x : a_n(x) \leq 1/2^{k+1}\} = \varnothing$. Let $U_1 = V_1$, and for each $k \geq 2$, let $U_k = V_k - V_{k-1}$. Then the U_k are clopen, pairwise disjoint, and their union is $\mathrm{coz}(a_n)$. Note that the IVT for monic polynomials holds in $C(U_k)$ for each k.

Let $k \in \mathbf{N}$. For $0 \leq i \leq n - 1$, let $b_i \in C(U_k)$, be defined by $b_i = a_i/a_n$. Define $q_k(t) \in C(U_k)[t]$ by $q_k(t) = t^n + b_{n-1}t^{n-1} + \cdots + b_0$. Note that if we restrict the polynomial $p(t)$ to U_k, then $p(t) = a_n q(t)$. Then $q_k(u) \geq 0$ and $q_k(v) \leq 0$ on U_k. So there exists a $w_k \in C(U_k)$ such that $u \wedge v \leq w_k \leq u \vee v$ on U_k and $q_k(w_k) = 0$.

Now define $w \in C(\mathrm{coz}(a_n))$ by $w(x) = w_k(x)$ whenever $x \in U_k$. Then $u \wedge v \leq w \leq u \vee v$. Since u,v must be bounded on $\mathrm{coz}(a_n)$, w is bounded on $\mathrm{coz}(a_n)$. Then since X is an F-space, there is a continuous extension w' of w to all of X with $u \wedge v \leq w' \leq u \vee v$. Then $p(w') = 0$ on $\mathrm{coz}(a_n)$. ❑

An f-ring A is said to have the *bounded inversion property* if $a > 1$ implies that a is invertible, or equivalently, if every maximal ideal of A is an ℓ-ideal (see [9]). For any f-ring, we let $\mathrm{Max}(A)$ denote the set of all maximal ideals of A, under the hull-kernel topology. If A satisfies the bounded inversion property, then $\mathrm{Max}(A)$ will be a compact Hausdorff space. Note that a 1-convex f-ring does satisfy the bounded inversion property; so for a 1-convex f-ring, $\mathrm{Max}(A)$ is a compact Hausdorff space. For a Tychonoff space X, $\mathrm{Max}(C(X)) \cong \beta X$. See [10] for more information on the hull-kernel topology.

The first part of the following theorem seems to be well known without appearing in the mathematical literature.

THEOREM 2.7: Suppose A is a semiprime f-ring with identity element.

(a) If A is 1-convex, then each maximal ideal of A contains a unique minimal prime ideal.

(b) If, in addition, A has the bounded inversion property, $\mathrm{Max}(A)$ is zero-dimensional, and A/P is an IVT ring for each minimal prime ideal P, then A is an IVT ring.

Before giving the proof of the theorem, we note the following.

REMARK 2.8: If $p(t) \in A[t]$, and $u,v \in A$ satisfy $p(u) > 0$ and $p(v) < 0$, then the zeros of the polynomials $p(t)$, $q(t) = p(t + (u \wedge v))$ are translates of each other by the amount $u \wedge v$, and if $u' = u - (u \wedge v)$, $v' = v - (u \wedge v)$, then $q(u') > 0$, $q(v') < 0$ and $u' \wedge v' = 0$. Similarly, if $p(u) < 0$ and $p(v) > 0$, or $p(u)p(v) < 0$.

Proof of Theorem2.7: (a) Suppose P and Q are distinct minimal prime ideals of A, and choose $x \in P^+ \backslash Q$. Since P is a minimal prime, there is a $y \in A^+ \backslash P$ such that $xy = 0$, and since Q is prime, it follows that $y \in Q$. By 1.2, the 1-convexity of A yields an r such that $|x - y| = r(x - y)$. Because A is a subdirect product of totally ordered integral domains and $x \wedge y = 0$, it follows easily that $r^+ y = 0$, whence $r^+ \in P$. Similarly, we have $(1 - r^+)x = 0$, and hence $(1 - r^+) \in Q$. Hence, $1 = r^+ + (1 - r^+) \in P + Q$, so $P + Q = A$. So no maximal ideal of A can contain two distinct minimal prime ideals. Because each maximal ideal of a ring with identity element must contain a minimal prime ideal, (a) holds.

(b) As noted above, A is a subdirect product of totally ordered integral domains. Suppose $p(t) = \sum_{i=0}^{n} a_i t^i \in A[t]$, $p(u) > 0$, and $p(v) < 0$ for some $u,v \in A$. By the preceding remark, we may assume $u \wedge v = 0$. By hypothesis, for each minimal prime ideal P, A/P is an IVT ring and so there is an $w_P \in A$ such that $0 \leq w_P \leq u \vee v$ and $p(w_P) \in P$. Now, let P^* denote the maximal ideal of A containing P. Since each maximal ideal contains a unique minimal prime ideal, there is, to each minimal prime ideal P, an $e_P \notin P^*$ such that $p(w_P)e_P = 0$. Since $\mathrm{Max}(A)$ is zero-dimensional, we may assume that each e_P is idempotent. Denoting $\{M \in \mathrm{Max}(A)\colon e_P \notin M\}$ by $\mathrm{coh}(e_P)$, it is clear that $\{\mathrm{coh}(e_P)\colon M \in \mathrm{Max}(A)\}$ is a cover of the compact space Max(A) by clopen sets. It has a finite subcover corresponding to idempotents which we will denote by $e_1, e_2, \ldots, e_n$. By a routine induction, we may assume that

(i) $e_ie_j = 0$ if $i \neq j$,

(ii) $\sum_{k=1}^{n} e_k = 1$,

(iii) $p(w_k)e_k = 0$ for $1 \leq k \leq n$.

For $1 \leq k \leq n$, let $p_k(t) = p(t) - a_0 + a_0e_k$. By (iii), $p_k(w_ke_k) = p(w_ke_k) - a_0 + a_0e_k = p(w_k)e_k = 0$ for each k. Let $w = \sum_{k=1}^{n} w_ke_k$, then $0 \leq w \leq (u \vee v)$. By (i) and (ii), $p(w) = \sum_{k=1}^{n} p_k(w_ke_k) = 0$. Hence, the IVT holds in A, and this completes the proof of the theorem. ❑

As a consequence, we can give another type of an IVT ring and an IVT space. An archimedean f-algebra is *Dedekind-MacNeille complete* if every subset with an upper bound has a supremum. We let $D(X)$ denote the set of all continuous functions mapping X to the extended real numbers, which are real valued on a dense subset of X. Then $D(X)$ is a distributive lattice under pointwise supremum and infimum, but in general not a group or a ring under pointwise operations. The space X is *extremely disconnected* if every open set has an open closure. If X is extremely disconnected (and in fact somewhat more generally), then $D(X)$ is an f-ring under pointwise operations. It is well known that every Dedekind-MacNeille complete f-algebra is isomorphic to a convex f-subring of $D(X)$ for some compact, extremely disconnected space X. See [4] for details.

It is shown in [5, Section 2, Proposition 6] that if $A = C(X)$ for some F-space X, then (in different terminology) the intermediate value theorem holds for polynomials in $(A/P)[t]$.

COROLLARY 2.9: (a) Every Dedekind-MacNeille complete f-algebra is an IVT ring.

(b) If X is a strongly zero-dimensional F-space, then$C^*(X)$ is an IVT ring. In particular, every compact zero-dimensional F-space is an IVT space.

Proof: (a) In light of Proposition 2.1(a), and the fact that every Dedekind-MacNeille complete f-algebra is isomorphic to a convex ℓ-subring of $D(X)$ for some compact, extremely disconnected space X, it suffices to show that if X is extremely disconnected, then $D(X)$ is an IVT ring. So suppose X is extremely disconnected, $p(t) = \sum_{i=0}^{n} a_it^i \in D(X)[t]$, and $p(u) > 0, p(v) < 0$ for some $u,v \in D(X)$. For any $f \in D(X)$, let $f^{-1}(\mathcal{R}) = \{x \in X : f(x) \in \mathcal{R}\}$. Each $f^{-1}(\mathcal{R})$ is a dense cozeroset of X and every extremely disconnected space is an F-space. It follows that each $f^{-1}(\mathcal{R})$ is C^*-embedded. Now let $Y = u^{-1}(\mathcal{R}) \cap v^{-1}(\mathcal{R}) \cap a_0{}^{-1}(\mathcal{R}) \cap a_1{}^{-1}(\mathcal{R}) \cap \cdots \cap a_n{}^{-1}(\mathcal{R})$. Then Y is a dense cozeroset of X. We may express Y as a countable disjoint union of compact open subsets of X, say, $Y = \bigcup_{n \in \mathbb{N}} K(n)$. Let $p_Y(t) \in C(Y)[t]$ (respectively, $p_{K(n)}(t) \in C(K(n))[t]$) denote the polynomial obtained from $p(t)$ by restricting all coefficient functions to Y (respectively, $K(n)$). Also, let u_n, v_n denote the respective restrictions of u,v to $K(n)$. Since $K(n)$ is compact and extremely disconnected and hence zero-dimensional [7, p.160], the previous theorem implies $K(n)$ is an IVT space. So for each n, there is $w_n \in C(K(n))$ such that $u_n \wedge v_n \leq w_n \leq u_n \vee v_n$ and $p_{K(n)}(w_n) = 0$. Since the $K(n)$ are clopen and $Y = \bigcup_{n \in \mathbb{N}} K(n)$, we may define $z \in C(Y)$ by $z(y) = w_n(y)$ when $y \in K(n)$. Now let z' be the unique extension of z to all of X. Then $p(z') = 0$ and $u \wedge v \leq z' \leq u \vee v$.

(b) Suppose X is a strongly zero-dimensional F-space and let $A = C(X)$. By the result mentioned just before the statement of the corollary, we know the IVT holds in A/P for each minimal prime ideal P. Note that $\mathrm{Max}(A) \cong \beta X = X$ if X is compact and hence $\mathrm{Max}(A)$ is zero-dimensional. Thus, by the previous theorem, X is an IVT space. ❑

Because every archimedean f-algebra is order-densely embedded in its Dedekind-MacNeille completion, we have:

COROLLARY 2.10: If A is an archimedean f-algebra, then there is an archimedean IVT ring B, which contains A as an f-subalgebra, and so that for each $0 < b \in B$, there is an $a \in A$ such that $0 < a \leq b$. The IVT ring B can be taken to be the Dedekind-MacNeille completion of A.

We say that a Tychonoff space X is a space of *dense constancy*, or a *DC-space*, if for each $f \in C(X)$ and each open subset V of X on which f is not identically zero, there is an open set W, contained in V so that $f|_W$ is constant and nonzero. An f-algebra A is a DC-algebra if for each $a \in A$ there is a maximal set of pairwise disjoint elements $\{c_i\}_{i \in I}$ and a set of real numbers $\{r_i\}_{i \in I}$ such that $ac_i = r_i c_i$. For a general treatment of DC-spaces and DC-algebras, we refer the reader to [3]. It is shown in 2.9 and 3.5 there that X is a DC-space if and only if $C(X)$ is a DC-algebra, and DC-algebras are both archimedean and have zero Jacobson radical.

If, in addition to being a DC-space, X also has the property that every nonempty open set contains a nonempty clopen set, then we say that X is a *Specker space*. A relatively simple example of a Specker space is described next. Consider $\alpha\mathbf{N}$, the one-point compactification of the natural numbers and $[0,1]$, the unit interval under the usual metric topology. Let X be the space obtained by refining the product topology on $\alpha\mathbf{N} \times [0,1]$ by making all points of the form (n,t), with $n \in \mathbf{N}$, $t \in [0,1]$, open. Then X is a Specker space that is not zero-dimensional.

The space Q of rational numbers is zero-dimensional but not Specker. In general, X is a Specker space if and only if βX is Specker, by 1.13(b) of [17].

Recall that an f-ring is said to be *laterally complete* if every subset of pairwise disjoint elements has a supremum. Every f-algebra has a *lateral completion*, which we will denote by A^L. That is, an f-algebra A may be embedded in a laterally complete f-algebra A^L so that no proper lattice-subalgebra of A^L containing A is laterally complete.

It is shown in [17] that a space X in which every nonempty open set contains a nonempty clopen set, is a Specker space if and only if $C(X) \subseteq S(X)^L$, where $S(X)$ denotes the subalgebra generated by all of the idempotents of $C(X)$. This is equivalent to saying that for each nonzero $f \in C(X)$, there is a nonempty clopen set C such that f_C is both constant and nonzero. To give the reader a better perspective, the elements of $S(X)$ are those continuous real-valued functions defined on X which can be expressed (uniquely) as a finite linear combination of characteristic functions of pairwise disjoint clopen subsets of X. The positive elements of $S(X)^L$ then are disjoint suprema of scalar multiples of characteristic functions of clopen sets.

Our last theorem of this section gives one additional type of IVT ring.

THEOREM 2.11: If X is a Specker space, then $S(X)^L$ is an IVT ring.

Proof: Let $p(t) = \sum_{i=0}^{n} a_i t^i \in S(X)^L[t]$, with $p(u) > 0$, and $p(v) < 0$, for some $u,v \in S(X)^L$. Suppose S_i (respectively, S_u, S_v) denotes a maximal family of pairwise disjoint clopen sets, with a_i (respectively, u,v) constant on each. Now let S denote the collection of all sets of the form $W_0 \cap W_1 \cap \cdots \cap W_n \cap W_u \cap W_v$, where $W_i \in S_i$, $W_u \in S_u$, and $W_v \in S_v$. Note that the members of S are pairwise disjoint clopen sets, and that S is maximal with respect to this property. For each $W \in S$, let $p_W(t) \in W[t]$ denote the polynomial obtained from $p(t)$ by restricting all coefficient functions to W. Let u_W, v_W denote the functions obtained by restricting u,v to W. Then $p_W(u_W) > 0$, $p_W(v_W) < 0$ and a_i, u, v are all constant on W. So $p_W(t)$ is effectively a real polynomial, which by the classical Intermediate Value Theorem has a root z_W between the minimum and maximum of the values of u_W and v_W. Now let $w_W \in S(X)$ be the function defined by $w_W(x) = z_W$ if $x \in W$ and $w_W = 0$ otherwise. Note that $p_W(w_W) = 0$. Now let $w = \bigvee_{W \in S} w_W$. Then $u \wedge v \le w \le u \vee v$ and $p(w) = 0$. ❑

3. IVT Spaces

It is an open question as to whether the converse of Corollary 2.9(b) holds; that is, whether a compact IVT space must be zero-dimensional. In fact, it is not known whether there exists any connected IVT space. In this section we show that the subspaces of any connected IVT space would have to have a certain disconnection property. To be more specific, we need the following definitions.

DEFINITIONS 3.1: A compact connected Hausdorff space is called a *continuum*; those with more than one point are called *nondegenerate*. If a continuum is the union of two proper subcontinua, it is called *decomposable*; otherwise it is called *indecomposable*. If every closed connected subspace of a continuum is indecomposable, it is called *hereditarily indecomposable*.

For more background, see [8], [12], [19], and [22]. We will show that any continuum contained in a compact IVT space is hereditarily indecomposable, and give examples of compact connected F-spaces that fail to be IVT spaces.

THEOREM 3.2: Every continuum contained in a compact IVT space is hereditarily indecomposable.

Proof: Suppose Y is a decomposable continuum contained in a compact IVT space X. Then $Y = Y_1 \cup Y_2$, where each Y_i is a proper subcontinuum. Pick $y_1 \in Y_1 \backslash Y_2$ and $y_2 \in Y_2 \backslash Y_1$. There is a cozeroset U containing y_1 such that $\mathrm{Cl}(U) \subseteq X \backslash Y_2$. Pick cozerosets V_i $(1 \le i \le 3)$ such that:

$$y_2 \in V_1 \subseteq \mathrm{Cl}(V_1) \subseteq V_2 \subseteq \mathrm{Cl}(V_2) \subseteq V_3 \subseteq X \backslash Y_1. \tag{1}$$

For $1 \le i \le 3$, choose $f_i \in C(X)^+$ such that:
(a) $\mathrm{coz} f_1 = U$, and $f_1 \le 1 = f_1(y_1)$,
(b) $f_2 \le .5 = f_2[V_2]$ and $f_2[X \backslash V_3] = 0$,
(c) $f_3 \le .25 = f_3[V_1]$ and $f_3[X \backslash V_2] = 0$.
Let $f = 2.5 + f_1 - f_2 - f_3$. The following facts are easily verified:
(i) $1.75 \le f \le 3.5 = f(y_1)$ and $f \ge 2.5$ on Y_1,

(ii) $f(y_2) = 1.75$ and $f \le 2.5$ on Y_2,

(iii) $f[Y_1 \cap Y_2] = 2.5$.

Finally, choose $h \in C(X)^+$ such that $Z(h) = \{x \in X: 2 \le f(x) \le 3\}$ and $h \le 1$.

Consider the polynomial $p(t) \in C(X)[t]$ given by: $p(t) = (t-1)(t-2)(t-3)[(t-f)^2 + h][(t-(f-1))^2 + h]$. Note that $p(f) = (f-1)(f-2)(f-3)h(h+1) > 0$. For, if $x \in f^{-1}[1.75, 2)$, then $p(f)(x) > 0$. If $x \in f^{-1}[2, 3]$, then since $h(x) = 0$, $p(f)(x) = 0$, while if $x \in f^{-1}(3, 3.5]$, then $p(f)(x) > 0$. Similarly, $p(f-1) = (f-2)(f-3)(f-4)(h+1)h < 0$. Because X is an IVT space, there is a $w \in C(X)$ such that $p(w) = 0$ and $f-1 \le w \le f$. It is routine to verify each of the following assertions:

(iv) If $3 < f(x) \le 3.5$, then $2 < w(x) \le 3.5$, so $w(x) = 3$.

(v) If $f(x) = 3$, then $2 \le w(x) \le 3$, so $w(x) = 2$ or 3.

(vi) If $2 < f(x) < 3$, then $1 < w(x) < 3$, so $w(x) = f(x)$ or $f(x) - 1$ or 2.

(vii) If $f(x) = 2$, then $1 \le w(x) \le 2$, so $w(x) = 1$ or 2.

(viii) If $1.75 \le f(x) < 2$, then $.75 \le w(x) < 2$, so $w(x) = 1$. Now let $C_1 = \{x \in Y_1: w(x) > 2\}$. Then C_1 is open in Y_1. But also, $C_1 = \{x \in Y_1: w(x) = f(x) \wedge 3\}$, so C_1 is clopen in the connected set Y_1. Since $y_1 \in C_1$, it follows that $C_1 = Y_1$.

Let $C_2 = \{x \in Y_2 : w(x) < 2\}$. Then C_2 is open in the connected set Y_2, and is nonempty since it contains y_2. Also, $C_2 = \{x \in Y_2 : w(x) = (f(x) - 1) \vee 1\}$, so $C_2 = Y_2$. Because Y is connected and $Y_1 \cap Y_2$ is nonempty, we have on this latter set, $w(x) = f(x) \wedge 3 = (f(x) - 1) \vee 1$. By (iii), $f(x) = 2.5$ on $Y_1 \cap Y_2$, so $w(x) = 2.5$ and 1.5. This contradiction completes the proof of the theorem. ❑

As is noted in [12, Section 48, V, Theorem 2], a continuum is indecomposable if and only if no proper subcontinuum has nonempty interior. Hence we have:

COROLLARY 3.3: No component of a compact IVT space contains a proper subcontinuum with nonempty interior.

Next we give examples of compact connected F-spaces that fail to be IVT spaces. First, we abbreviate $\beta X \backslash X$ by X^*, and the subspace $[0,\infty)$ of $\mathcal{R}$ by H. We will make use of the following results due to R.G. Woods and D.P. Bellamy; see [22, 9.13 and 9.33] for more explicit references.

(a) H* is an F-space and an indecomposable continuum that is not hereditarily indecomposable.

(b) If $n > 1$, then $(\mathcal{R}^n)^*$ is an F-space and a decomposable continuum.

Recall also that if X is locally compact and σ-compact, then X^* is an F-space; see [7, 14.27].

EXAMPLES 3.4: H* and $(\mathrm{R}^n)^*$ are compact connected F-spaces that fail to be IVT.

We will call a space X *nice* if each of its nondegenerate components contains a decomposable continuum. Then we have:

COROLLARY 3.5: Every nice compact IVT space is zero-dimensional.

As the following example shows, there are compact connected F-spaces that fail to be nice. Whether there are compact IVT spaces that are not nice remains an open question.

EXAMPLE 3.6: Theorem 7 of [21] says that if X is the union of countably many disjoint hereditarily indecomposable continua, then every component of X^* is hereditarily indecomposable and X^* contains a nondegenerate hereditarily indecomposable continuum. Also, because X is locally compact and σ-compact, X^* is an F-space. As noted in the notes in [8, Section 5], it follows easily from this that if Y is the union of a discrete collection of pseudoarcs in the plane then Y^* is an hereditarily indecomposable subcontinuum of $(\mathcal{R}^2)^*$. (See [19] for the definition of a pseudoarc.)

4. ALTERNATIVE VERSIONS OF THE INTERMEDIATE VALUE THEOREM

In this section, we look at two variations on IVT spaces and rings.

DEFINITIONS 4.1: Suppose A is a commutative f-ring with identity element, let $A[t]$ denote the ring of polynomials with coefficients in A, and let G denote a subset of $A[t]$ closed under translation by elements of A.

(1) If, for each $p(t) \in G$ for which $p(u)p(v) \leq 0$ for some $u,v \in A$, there is a $w \in A$ such that $u \wedge v \leq w \leq u \vee v$ and $p(w) = 0$, we say that the *strong intermediate value theorem holds for* G. In case $G = A[t]$, then A is called a *strong IVT ring*.

(2) If for some Tychonoff space X, $C(X)$ is a strong IVT ring, then X is called a *strong IVT space*.

It is easy to see that if an f-ring satisfies the strong intermediate value theorem for a certain subset G of polynomials, then it also satisfies the intermediate value theorem for G. In particular, every strong IVT ring (space) is an IVT ring (space). We will see that in general, the intermediate value theorem for a subset G of polynomials and the strong intermediate value theorem for G are not equivalent. For linear polynomials, however, they are.

THEOREM 4.2: Suppose A is a commutative semiprime f-ring with identity element. In A, the intermediate value theorem holds for linear polynomials if and only if the strong intermediate value theorem holds for linear polynomials.

Proof: We need only show that if the IVT for linear polynomials holds, then the strong IVT for linear polynomials holds. So suppose the IVT for linear polynomials holds in A, $p(t) = a + bt \in A[t]$, and $p(u)p(v) \leq 0$ for some $u,v \in A$. As noted in Remark 2.8, we may assume that $u \wedge v = 0$.

By 1.2, there is an $r \in A$ such that $b = r|b|$ and $|r| \leq 1$. If $f = r^+(u \vee v)$ and $g = r^-(u \vee v)$, then $p(f)p(g) = a^2 + ab(r^+(u \vee v) + r^-(u \vee v)) = a^2 + ab|r|\,(u \vee v) = a^2 + ab(u \vee v) = p(u)p(v)$. Hence:

$$p(f)p(g) \leq 0. \tag{*}$$

We will show next that:

$$p(f) \geq 0 \text{ and } p(g) \leq 0. \tag{**}$$

By 1.1, A is a subdirect product of totally ordered prime rings A_x for some index set X. For any $a \in A$, we let a_x denote the projection of a in A_x. If $b_x \geq 0$, then $r_x^- = 0$, so $p(f)_x = a_x + b_x r_x^+(u_x \vee v_x)$ and $p(g)_x = a_x$. Thus, if $p(f)_x < 0$, then $a_x = p(g)_x < 0$ as well, contrary to (*). If $b_x < 0$, then $r_x^+ = 0$, $r_x^- = 1$, $p(f)_x = a_x$ and $p(g)_x = a_x + b_x(u_x \vee v_x)$. Hence, if $p(f)_x < 0$, this inequality holds for $p(g)_x$ as well, contrary to (*). It follows that $p(f) \geq 0$ and a symmetric argument shows that $p(g) \leq 0$ and (**) holds. Since the IVT for linear polynomials holds, there is a $w \in A$ such that $p(w) = 0$ and $0 = f \wedge g \leq w \leq f \vee g = |r|(u \vee v) \leq u \vee v$. ❑

Recall that a *Lindelöf space* is one for which every open cover has a countable subcover. For a Lindelöf space, we are able to give a characterization of spaces in which the strong IVT holds for linear and monic quadratic polynomials.

THEOREM 4.3: An F-space is strongly zero-dimensional if and only if it satisfies the IVT for linear and the strong IVT for monic quadratic polynomials.

Proof: $\Rightarrow$ By Theorem 2.2, the IVT holds for linear polynomials.

Next we show that the strong IVT for monic quadratic polynomials holds in $C(X)$. Suppose $p(t) \in C(X)[t]$ and $u,v \in C(X)$ such that $p(u)p(v) \leq 0$. We may, by completing the square, assume $p(t)$ is of the form $p(t) = t^2 - f$ for some $f \in A$. We may also assume that $f \neq 0$. Now $p(u)p(v) = (u^2 - f)(v^2 - f) = u^2v^2 - fv^2 - fu^2 + f^2 \leq 0$. So $0 \leq u^2v^2 + f^2 \leq f(u^2 + v^2)$. It follows that $f \geq 0$, and there is a $b \in A^+$ such that $f = b^2$. Thus $p(t) = t^2 - b^2$. Let $U_1 = \{x \in X\colon b(x) > u(x) \vee v(x)\}$ and let $U_2 = \{x \in X\colon -b(x) < u(x) \wedge v(x) \text{ or } -b(x) > u(x) \vee v(x)\}$. Then U_1 and U_2 are disjoint cozerosets of the F-space X, and hence there are disjoint zerosets Z_1,Z_2 such that $U_1 \subseteq Z_1$ and $U_2 \subseteq Z_2$. Because X is strongly zero-dimensional, there is a clopen set W such that $Z_1 \subset W$ and $Z_2 \cap W = \varnothing$ (see [7, Chapter 16]). Define $w \in C(X)$ by letting $w(x) = -b$ if $x \in W$ and $w(x) = b$ otherwise. Then $u \wedge v \leq w \leq u \vee v$ and $p(w) = 0$.

$\Leftarrow$ Assume X is an F-space which satisfies the IVT for linear and the strong IVT for monic quadratic polynomials. By [7, Theorem 16.17] it suffices to show that any two disjoint zerosets are separated by a partition. So suppose Z_1 and Z_2 are disjoint zerosets. Choose $f_1, f_2 \in C(X)^+$ such that $f_1, f_2 \leq 1$, $Z(f_1) = Z_1$ and $Z(f_2) = Z_2$. Let $f = 3 + f_1 - \phi_2$ and let $p(t) = (t-3)(t-1)$. Then $p(f)p(f-2) = (f-1)(f-3)^2(f-5) \leq 0$ since $2 \leq f \leq 4$. By the strong IVT for monic quadratic polynomials, there is a $w \in C(X)$ such that $f - 2 \leq w \leq f$ and $p(w) = 0$. Then if $x \in Z_1$, $w(x) = 1$ and if $x \in Z_2$, $w(x) = 3$, elsewhere $w(x) = 1$ or 3. Hence $\{x \in X : w(x) < 2\}$ is a clopen set containing Z_1 and disjoint from Z_2. So X is zero-dimensional. ❑

It is now easy to see that the IVT for monic quadratic polynomials is not equivalent to the strong IVT for monic quadratic polynomials. Indeed, by Theorem 2.3(b), for any space X, the IVT for monic quadratic polynomials holds, whereas, by the previous theorem, the strong IVT for monic quadratic polynomials does not hold for infinite connected F-spaces.

It follows immediately from the previous theorem that:

COROLLARY 4.4: A compact zero-dimensional F-space is a strong IVT space for linear and monic quadratic polynomials.

We now turn to a second variation on the definition of the intermediate value theorem holding for a subset of an f-ring.

DEFINITIONS 4.5: Suppose A is a commutative f-ring with identity element, let $A[t]$ denote the ring of polynomials with coefficients in A, and let G denote a subset of $A[t]$, closed under translation by elements of A.

(1) If, for each $p(t) \in G$ for which $p(u) > 0$ and $p(v) < 0$ for some $u,v \in A$, there is a finite subset $\{w_1, w_2, \ldots, w_m\} \subseteq A$ such that $u \wedge v \leq w_i \leq u \vee v$ for each i and $p(w_1)p(w_2) \cdots p(w_m) = 0$, we say that the *weak intermediate value theorem (weak IVT) holds for G*. In case $G = A[t]$, then A is called a *weak IVT ring*.

(2) If for some Tychonoff space X, $C(X)$ is a weak IVT ring, then X is called a *weak IVT space*.

Suppose n is a positive integer. An f-ring is said to have *rank n* if there are no more than n minimal prime ideals contained in each maximal ℓ-ideal and some maximal ℓ-ideal contains n minimal prime ideals, and is said to have *finite rank* if it has rank n for some positive integer n. A *valuation domain* is a domain in which for any two elements, one is a multiple of the other and an f-ring is called an *SV-ring* if A/P is a valuation domain for each prime ideal P of A. Note that any 1-convex f-ring has rank *1* and is an SV-ring. Any uniformly complete SV-ring has finite rank, as is shown in [11, 4.1]. It is noted in [11] that an f-ring A with identity element and bounded inversion is an SV-ring if and only if for any (minimal) prime ideal P of A, A/P is 1-convex. For f-rings with finite rank and bounded inversion, the weak IVT for linear polynomials has a characterization analogous to that of the IVT for linear polynomials.

If M is a maximal ideal of a commutative semiprime ring A with identity, let O_M denote the set of $a \in A$ for which there is some $b \notin M$ such that $ab = 0$. It is well known that O_M is the intersection of all the minimal prime ideals of A that are contained in M.

THEOREM 4.6: Suppose A is a commutative semiprime f-ring with identity element and bounded inversion.

(a) If the weak IVT for linear polynomials holds, then A is an SV-ring.

(b) If A has finite rank and is an SV ring, then the weak IVT for linear polynomials holds.

Proof: (a): Suppose $0 \leq a + P \leq b + P$ in A/P for some prime ideal P of A. Then there is $p_1, p_2 \in P^+$ such that $0 \leq a + p_1 \leq b + p_2$. Let $p(t) = a + p_1 + (b + p_2)t \in A[t]$. Then $p(0) = a + p_1 \geq 0$ and $p(-1) = a + p_1 - (b + p_2) \leq 0$. Because A is weak IVT for linear polynomials, there is a finite subset $\{w_1, w_2, \ldots, w_m\} \subseteq$ A such that $-1 \leq w_i \leq 0$ for each i and $p(w_1)p(w_2) \cdots p(w_m) = 0$. Since P is prime, $p(w_i) \in P$ for some i. Hence, $a + bw_i \in P$. Thus $a = -w_i b$ mod$P$ and A/P is 1-convex. So A is an SV-ring.

(b) Suppose $p(t) = a + bt \in A[t]$, $p(u) > 0$, and $p(v) < 0$. Since A is closed under bounded inversion, every maximal ideal of A is an ℓ-ideal. We will show next that:

$$\text{if } M \text{ is a maximal ideal of } A\text{, there is a finite subset } \{y_1, y_2, \ldots, y_k\} \subseteq A \text{ such that } u \wedge v \leq y_i \leq u \vee v \text{ and } z_M = p(y_1)p(y_2) \cdots p(y_k) \in O_M. \qquad (**)$$

Now by hypothesis, M contains only finitely many minimal prime ideals $\{P_1, \ldots, P_k\}$. For $1 \le i \le k$, let $p_i(t) = (a + P_i) + (b + P_i)t \in (A/P_i)[t]$ and note that $p_i(u) \ge 0$ and $p_i(v) \le 0$ in A/P_i. For each i, A/P_i is 1-convex since A is an SV-ring. By Theorem 2.2, there are for $1 \le i \le k$, $y_i \in A$ with $u \wedge v \le y_i \le u \vee v \bmod P_i$ and $p(y_i) = 0 \bmod P_i$. For each i, let $y_i' = (y_i \vee (u \wedge v) \wedge (u \vee v))$. Then $u \wedge v \le y_i' \le u \vee v$ and $z_M = p(y_1')p(y_2') \cdots p(y_k') \in \cap_{i=1}^{k} P_i = O_M$. Hence (**) holds.

For each $M \in \mathrm{Max}(A)$, there is a $z_M' \notin M$ such that $z_M' > 0$ and $z_M z_M' = 0$. Letting $\mathrm{coh}(z_M') = \{M \in \mathrm{Max}(A): z_M' \notin M\}$, we see that $\{\mathrm{coh}(z_M'): M \in \mathrm{Max}(A)\}$ is an open cover of the compact space $\mathrm{Max}(A)$. So there is a finite set $\{M_1, \ldots, M_n\} \subseteq \mathrm{Max}(A)$ such that $\mathrm{Max}(A) = \cup_{i=1}^{n} \mathrm{coh}(z_{M_i}')$. Then $z = \sum_{i=1}^{n} z_{M_i}'$ is in no maximal ideal of A and hence is invertible. But $z z_{M_1} z_{M_2} \cdots z_{M_n} = 0$ and so $z_{M_1} z_{M_2} \cdots z_{M_n} = 0$. ❑

An argument almost identical to the proof of part (*b*) of the previous theorem shows the following.

THEOREM 4.7: Suppose A is a semiprime f-ring with identity element. If A is 1-convex and A/P is an IVT ring for each minimal prime ideal of P, then A is a weak IVT ring.

NOTE ADDED IN PROOF, MARCH 14, 1996: A. Dow and K.P. Hart announce that a resricted form of the IVT, namely for monic polynomials that can be written as

$$\prod_{i=1}^{n} (t - f_i)$$

characterizes hereditary indecomposability for compact F-spaces.

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Countably Compact Spaces which Cannot Be Cancellative Topological Semigroups

SANJU HOTA,[a,d] S. PURISCH,[b] AND M. RAJAGOPALAN[c,d]

[a,c]*Department of Mathematics*
Tennessee State University
Nashville, Tennessee 37209
and
[b]*Department of Mathematics*
Barry University
Miami Shores, Florida 33161

ABSTRACT: Forty years ago A.D. Wallace asked whether a countably compact, cancellative topological semigroup is a topological group.

We will prove that for various types of spaces Wallace's problem has an affirmative answer. This is done by showing that the most promising candidates for counterexamples do not admit cancellative topological semigroup structures.

1. INTRODUCTION

In 1955, Wallace [19] asked whether every countably compact, cancellative topological semigroup is a topological group. In 1994, Robbie and Svetlichny gave a counterexample under the continuum hypothesis. The construction can be done under Martin's Axiom as well [18]. This still leaves open the question whether an affirmative answer to Wallace's question is consistent in ZFC and for what type of spaces is there a positive answer.

On the affirmative side, in 1952, Numakura [12] showed that every compact, cancellative topological semigroup is a topological group. Ellis [5], [6] showed in 1957 that a locally compact group G with separately continuous multiplication (i.e., the multiplication restricted to $\{p\} \times G$ and to $G \times \{p\}$ are continuous for all $p \in G$) is a topological group. Brand [1] and Pfister [13] proved that Ellis' result remains true for completely regular spaces with continuous multiplication if either "Cech complete" or "locally countably compact" replaces "locally compact." In 1972 Mukherjea and Tserpes [11] proved that every countably compact, cancellative topological semigroup which is first countable is a topological group. D.L. Grant [9] in 1993 extended this result for the completely regular case by showing that every sequentially compact, cancellative topological semigroup is a topological group.

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We will give further results on the affirmative side of Wallace's problem. For example a Baire space is a topological group if it is a torsion group with continuous multiplication. Also, no subset of βN which is locally compact, countably compact and infinite can admit a cancellative topological semigroup structure. In addition we look at cancellative topological semigroups either with a countably compact square or where the monothetic subsemigroups are groups.

Besides the consistency of Wallace's problem, related questions arise. Watson [9] asked whether a pseudocompact, first countable, cancellative topological semigroup is a topological group. Using convergence properties of cancellative topological semigroups, we show that the obvious types of candidates for counterexamples to Watson's question do not admit a cancellative topological semigroup structure. For example, the ψ spaces of Isbell and certain expansions of a compact first countable topology do not admit such a structure. We also show that variations of the Tychonoff plank (which are pseudocompact, not countably compact, and not first countable) do not admit a cancellative topological semigroup structure.

Note that "cancellative" is essential since for any space X and a fixed $p \in X$, the multiplication $xy = p$ for all $x, y \in X$ gives X a topological semigroup structure.

In this paper all spaces are Hausdorff (except in the hypothesis of Lemma 3.1, which is clear from the context). We follow the notation in [4] for topological semigroups and [7] for topological spaces.

A *paratopological group* is a topological semigroup which is algebraically a group. If S is a semigroup and $x \in S$, then $\theta(x) = \{x, x^2, x^3, \ldots\}$ is the Abelian subsemigroup generated by x, and if S is topological semigroup then $\Gamma(x) = \mathrm{cl}(\theta(x))$ ($\mathrm{cl}(A)$ denotes the closure of A for all $A \subset S$) is also a subsemigroup of S. S is called monothetic if $S = \Gamma(p)$ for some $p \in S$. Let N denote the natural numbers. As a topological space, give N the discrete topology. Let βX denote the Stone-Čech compactification of a space X.

2. TORSION GROUPS AND COUNTABLY COMPACT SEMIGROUPS

A group G is a *torsion* group if $\theta(x)$ is finite for all $x \in G$.

THEOREM 2.1: Let G be a paratopological torsion group which is a Baire space. Then G is a topological group.

Proof: We only need to show that the inverse function is continuous. So let $F_n = \{x \in G : x^n = e\}$ for every $n \in N$. Then each F_n is closed and $\bigcup_{n \in N} F_n = G$. Since G is a Baire space, some F_k has nonempty interior. Now $x^k = e$ for each $x \in F_k$. So $x^{-1} = x^{k-1}$ for all $x \in F_k$. Since multiplication is continuous, we have that the function $x \to x^{k-1}$ is continuous on F_k. Since F_k has a nonempty interior, the inverse function is continuous at some point of G. Thus, the inverse function is continuous at every point of G (since, if the inverse function is continuous at $g \in G$ and $\{x_\alpha\}_{\alpha \in A}$ is a net converging to $x \in G$, then $\{x_\alpha{}^{-1}\}_{\alpha \in A}$ converges to $yg^{-1} = x^{-1}$ where $y \in G$ such that $xy = g$). ❑

LEMMA 2.2: [4, Corollary 3.3] Let $S = \Gamma(p)$ be a compact monothetic semigroup. If p is not isolated, then S is a topological group.

CONJECTURE 2.3:[1] A countably compact subsemigroup of a compact monothetic group is a group.

THEOREM 2.4: Let S be a completely regular, countably compact, cancellative topological semigroup such that $S \times S$ is pseudocompact. Then S is a topological group provided that Conjecture 2.3 is true and $\theta(p)$ is dense in itself or finite for each $p \in S$.

Proof: By Glicksberg's Theorem, $\beta(S \times S) = \beta S \times \beta S$. Now the multiplication $(x,y) \rightarrow xy$ from $S \times S$ to βS extends to a continuous function from $\beta(S \times S)$ to βS. Since $\beta(S \times S) = \beta S \times \beta S$ we have that the associative multiplication on S extends continuously to an associative multiplication on βS. Hence, βS is a compact topological semigroup under this multiplication.

Let $x \in S$. If $\theta(x)$ is finite, then $\theta(x)$ is a compact, monothetic, topological group. So assume $\theta(x)$ is infinite. Then $\Gamma_{\beta S}(x)$ (the closure in βS of $\theta(x)$) is a compact semigroup. So by the hypothesis and Lemma 2.2, $\Gamma_{\beta S}(x)$ is a compact, monothetic, topological group. In any case since $F = S \cap \Gamma_{\beta S}(x)$ is a countably compact subsemigroup of $\Gamma_{\beta S}(x)$, F is a group by Conjecture 2.3. Hence, the identity u of F is in S. Since S is a cancellative semigroup and u is an idempotent (i.e., $u^2 = u$), u is the identity of S. Furthermore, the inverse of x in F is in S. Since x is arbitrary, we have that S is a countably compact, paratopological group. So βS is also a group. So by Ellis' theorem [5], [6] βS is a topological group. Hence, the inverse function is continuous. ❑

NOTE 2.5: In [8] D.L. Grant proved the following results complementing Theorem 2.4. Let G be a pseudocompact, completely regular paratopological group. Then G is a topological group if and only if $G \times G$ is pseudocompact. Also, in [15] Reznichenko extended Grant's result showing that a pseudocompact, completely regular paratopological group is a topological group. In light of Reznichenko's result the referee asked whether in Theorem 2.4 the hypothesis could be dropped. The authors came close to proving this and feel it can be done.

PROPOSITION 2.6: Let S be a cancellative (countably compact) topological semigroup. If every monothetic subsemigroup of S is a group, then S is a (topological) group. Hence, any counterexample to Wallace's problem contains a monothetic counterexample.

Proof: Let $x \in S$. As in the proof of Theorem 2.4, since by hypothesis $\Gamma(x)$ is a group, the identity of $\Gamma(x)$ is the identity of S, and $x^{-1} \in \Gamma(x) \subset S$. So S is a paratopological group. Hence, if S is countably compact, then S is a topological group by [1] and [13]. ❑

LEMMA 2.7: [4, Theorem 3.2] The set of cluster points of a compact monothetic semigroup form a group.

LEMMA 2.8: [2] The closure in βN of any countable subset is extremely disconnected.

[1]Robbie recently pointed out that under CH, the example in [16] can be used to construct a counterexample to Conjecture 2.3.

LEMMA 2.9: [14] No locally compact, nondiscrete, topological group is extremely disconnected.

THEOREM 2.10: Let X be a locally compact, countably compact, infinite subset of βN. Then X does not admit a cancellative topological semigroup structure.

Proof: Suppose that X had a cancellative topological semigroup structure. Let $b \in X$, and let closures be taken in X. Since $\Gamma(b)$ is locally compact and countably compact, its square is countably compact, (e.g., see [7]). Then as in Theorem 2.4, Glicksberg's theorem extends the multiplication on $\Gamma(b)$ to a continuous associative multiplication on $\beta(\Gamma(b))$.

First suppose that $\theta(b)$ were infinite and b^m were a boundary point of $\theta(b)$ for some $m \in N$. Then there is a net $\{b^{n_\alpha}\}_{\alpha \in A}$ converging to b^m where each $n_\alpha \in N$. Hence, $b^{j+n_\alpha} = b^j b^{n_\alpha} \to b^{j+m}$, and so all but at most finitely many elements of $\theta(b)$ are cluster points of $\theta(b)$. Hence, the set D of cluster points of $\beta(\Gamma(b))$ consists of all but at most finitely many elements of $\beta(\Gamma(b))$. Now $\beta(\Gamma(b))$ is extremely disconnected since by Lemma 2.8 $\Gamma(b)$ is. Therefore, D is extremely disconnected, infinite, and compact. But D is a topological group by Lemma 2.7. This contradicts Lemma 2.9.

Now suppose that $\theta(b)$ were infinite and discrete. Then the closure in βN of $\theta(b)$ is homeomorphic to βN. Hence, the set of cluster points of $\beta(\Gamma(b))$ is homeomorphic to the nonhomogeneous space $\beta N - N$. This contradicts Lemma 2.7.

So $\theta(b)$ is finite for each $b \in X$. Hence, X is a paratopological group and therefore a topological group by [5], [6]. Let A be a countably infinite subset of X, and G be the closure of the subgroup of X generated by G. Then G is separable and hence extremely disconnected by Lemma 2.8. Again, this contradicts Lemma 2.9 since G is a nondiscrete, locally compact, topological group. ❑

The last paragraph of the proof of Theorem 2.10 also proves the following.

PROPOSITION 2.11: If X is a locally compact subset of βN and $p \in X$ such that p is a cluster point of a countable subset of X, then X cannot be a topological group.

QUESTION 2.12: Are there any countably compact infinite subsets of βN that admit a cancellative topological semigroup structure? What if "not discrete" replaces "countably compact"?

3. PSEUDOCOMPACT, NOT COUNTABLY COMPACT SPACES

In this section we investigate Watson's question [9]: Is every pseudocompact, first countable, cancellative topological semigroup a topological group? Since the question has an affirmative answer for countably compact spaces [11], we look at spaces which are pseudocompact but not countably compact. Three fundamental types of such spaces are the Ψ spaces of Isbell, an expansion of the topology of a compact space, and variation of the Tychonoff plank. It turns out that none of them admit a cancellative topological semigroup structure.

Let (X,τ) be a topological space and $D \subset X$. A *simple extension of* τ *by* D is the topology τ_1 of X generated by $\tau \cup \{D\}$. Thus, a subset U of X is a τ_1-neighborhood of p where $p \in X$ iff U is a τ-neighborhood of p or $p \in W \cap D \subset U$ for some $W \in \tau$. Let S be a topological space with a multiplication m: $S \times S \to S$ which makes S algebraically a semigroup. If the restrictions $m|(\{p\} \times S)$ and $m|(S \times \{p\})$ are continuous for all $p \in S$, then m is called a *separately continuous multiplication.*

LEMMA 3.1: (See [3, Theorem 1.6].) Every T_0 topological group is Hausdorff and completely regular.

LEMMA 3.2: Let S be a compact space with a separately continuous multiplication that makes S algebraically a semigroup. Then S contains an idempotent.

Proof: The proof is just the proof of Theorem 1.8 in [4]. This theorem states that a compact topological semigroup S contains an idempotent. The only multiplication used in the proof and the results on which the proof rely are of the form pT and Tp for $p \in S$ and $T \subset S$. So only separately continuous multiplication is needed. ❑

THEOREM 3.3: Let (X,τ) be a compact space, D be a dense and codense subset of X, and τ_1 be the simple extension of τ by D. Then (X,τ_1) is Hausdorff, pseudocompact, not regular, and does not admit a cancellative topological semigroup structure. Moreover, if (X,τ) is first countable, then (X,τ_1) is first countable and not countably compact.

Proof: It is routine to verify that if f is a function from X to the real line R, then f is continuous in the topology τ if and only if f is continuous in τ_1. So (X,τ_1) is pseudocompact. Since there are no disjoint τ_1-open sets separating p and $X-D$ for $p \in D$, (X,τ_1) is not regular. The last remark of the theorem and the failure of countable compactness of (X,τ_1) follow easily.

Suppose (X, τ_1) were a cancellative topological semigroup. For $p \in X$, define the functions l_p and r_p from X to X by $l_p(x) = px$ and $rp(x) = xp$ for all $x \in X$. Then l_p: $(X,\tau_1) \to (X,\tau_1)$ is continuous. Now if f: $(X,\tau) \to R$ is continuous, then f: $(X,\tau_1) \to R$ is continuous. Hence, $f \circ l_p$: $(X,\tau_1) \to R$ and similarly $f \circ l_p$: $(X,\tau) \to R$ are continuous for all $p \in X$. Since this is true for all $f \in C(X,\tau)$ (the collection of the real valued continuous functions on (X,τ)) and (X,τ) is compact, we get that l_p: $(X,\tau) \to (X,\tau)$ is continuous for all $p \in X$. Similarly r_p: $(X,\tau) \to (X,\tau)$ is continuous for $p \in X$. That is , the multiplication on (X,τ) is separately continuous.

Let $p \in X$ and $\Gamma(p)$ be the τ closure of $\theta(p)$. Then with respect to the multiplication on X, the semigroup $\Gamma(p)$ has an idempotent by Lemma 3.2. Since X is algebraically a cancellative semigroup, this idempotent is identity e of X. If $e = p^n$ for $n \in N$, then p^{-1} is e or p^{n-1} . If $e \notin \theta(p)$, then there is a net $\{p^{n_\alpha}\}_{\alpha \in A}$ with each n_α an integer greater than 1 and $p^{n_\alpha} \to e$ in (X,τ). Let $k_\alpha = n_\alpha - 1$. Then the net $\{p^{k_\alpha}\}_{\alpha \in A}$ has a subnet $\{p^{k_{\alpha\beta}}\}_{\beta \in B}$ which converges to some u in the compact space (X,τ). Then $p \cdot p^{k_{\alpha\beta}} \to pu$ and $p \cdot p^{k_{\alpha\beta}} = p^{n_{\alpha\beta}} \to e$. Hence $pu = e$. Similarly, $up = e$. That is, $u = p^{-1}$. Since p is arbitrary, X is algebraically a group. Therefore, by Ellis' result [5], [6] , (X,τ) is a topological group.

We claim that the inverse function on (X,τ_1) is continuous. To see this fix a point $p \in X$, such that $q = p^{-1} \in X - D$. Let U be a τ_1-neighborhood of q and hence a τ-neighborhood of q. Since the inverse function is continuous on (X,τ) there is a τ-open neighborhood V of p such that $V^{-1} \subset U$. Since $\tau \subset \tau_1$ it follows that $V \in \tau_1$. Hence, the inverse function on (X, τ_1) is continuous at p. Now let $x \in X$ and $\{x_\alpha\}_{\alpha \in A}$ be a net in (X, τ_1) converging to x. Then $px^{-1}x_\alpha \to p$ in (X, τ_1); so $x_\alpha{}^{-1}xp^{-1} \to p^{-1}$ in (X, τ_1) by the continuity of the inverse function at p. Hence, $x_\alpha{}^{-1} \to x^{-1}$ in (X, τ_1), that is, the inverse function on (X, τ_1) is continuous.

So we get that (X, τ) is a topological group which is Hausdorff and not regular. This contradicts Lemma 3.1. Therefore, (X, τ_1) does not admit a topological semigroup structure. ❑

NOTE 3.4: As in the proof of Theorem 3.3, nets often play a key role in proofs concerning topological semigroups. We want to consider convergence properties at a point, e.g., the type of nets that can converge to a point. So we define two nets $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\alpha\}_{\delta \in D}$ to be of the same *net type* if there is an order-preserving isomorphism $\sigma: A \to D$ (i.e., σ is a bijection and $\sigma(\alpha) < \sigma(\beta)$ if and only if $\alpha < \beta$ for all $\alpha, \beta \in A$) such that $y_{\sigma(\alpha)} = y_{\sigma(\beta)}$ iff $x_\alpha = x_\beta$ for all $\alpha, \beta \in A$. Hence, given a cancellative topological semigroup S, elements x and y of S, and a net converging to x, there are nets of the same type converging to xy and yx. In particular if S has an identity element e, for any net converging to e there are nets of the same type converging to each element of S. Also, for example, in S an isolated point cannot have a nonisolated factor, a point with a countable neighborhood base cannot have a factor that is the limit of an uncountable transfinite sequence which is not eventually constant, and a point which is not the limit of a (nontrivial, i.e., not eventually constant) sequence cannot have a factor that is.

NOTE 3.5: A family F of subsets of a set is *almost disjoint* if $\alpha \cap \beta$ is finite for all $\alpha, \beta \in F$. A ψ *space* of Isbell (see [10]) is a set $X = N \cup D$ such that: (1) D is a maximal almost disjoint family of infinite subsets of N, (2) the points of N are isolated, and (3) if $\alpha \in D$, then a neighborhood base of α in X consists of all sets of the form $\{\alpha\} \cup B$ where $B \subset \alpha$ and $\alpha - B$ is finite. Any Ψ space is locally compact, Hausdorff, first countable, pseudocompact, and not countably compact. Each of its discrete clopen subsets is finite, N is dense in X, and D is discrete and closed.

THEOREM 3.6: Let $X = Y \cup Z$ be a space such that Y and Z are nonempty discrete subsets of X, and Y is dense and open in X. Then X does not admit a cancellative topological semigroup structure. In particular, no Ψ space admits a cancellative topological semigroup structure.

Proof: Suppose X did admit a cancellative topological semigroup structure. For all $x \in X$ and $z \in Z$, $xz \in Z$ by Note 3.4 since the points of Y are isolated while the points of Z are not. Now fix $z \in Z$. Since Y is dense in X there is a net $\{y_\alpha\}_{\alpha \in A}$ in Y such that $y_\alpha \to z$. So $\{y_\alpha z\}_{\alpha \in A}$ is a (not eventually constant) net in Z converging to $z^2 \in Z$, contradicting Z being discrete. ❑

Using Note 3.4, the proof of Theorem 3.6 translates into a proof of the following result.

PROPOSITION 3.7: Let $X = Y \cup Z$ be a space such that there exists $z \in Z$ and a net in Y converging to z such that no net of this type converges to any point of Y, and no net of this type contained in Z converges to any point of Z. Then X does not admit a topological semigroup structure.

DEFINITION 3.8: Let γ be an ordinal and consider the ordinal $\gamma+1$ with the usual order topology. So $\gamma+1$ is a compact space. For all $\alpha \in \gamma$ let X_α be a topological space, and $p_\alpha \neq q_\alpha \in X_\alpha$. Let Y be the disjoint union $(\gamma+1) \oplus (\oplus \{X_\alpha : \alpha \in \gamma\})$ and give Y the direct sum topology of the spaces $\gamma+1$ and the X_α's. Let X be the quotient space obtained by identifying p_α with α and q_α with $\alpha+1$ for all $\alpha \in \gamma$. Call X a *long line-like space* using γ and the spaces X_α.

Recall that ω is the first infinite ordinal and ω_1 is the first uncountable ordinal.

NOTE 3.9: $\gamma = \omega_1$, each $X_\alpha = [0,1]$, $p_\alpha = 0$ and $q_\alpha = 1$, then the long-line like space obtained from ω_1, and the spaces X_α is the usual long line.

THEOREM 3.10: Let γ be an ordinal of uncountable cofinality and X_α be a compact first countable space for each $\alpha \in \gamma$. Let $Y = (X \times (\omega+1)) - \{\langle \gamma, \omega \rangle\}$ where X is a long-line like space using γ and the X_α's. Then Y is pseudocompact, locally compact, and not countably compact. Moreover, Y does not admit a cancellative topological semigroup structure.

Proof: The proof that Y is pseudocompact, locally compact, and not countably compact is analogous to the proof (e.g., see [10] and [17]) that the Tychonoff plank has those properties.

Now suppose that Y admits a cancellative topological semigroup structure. Then $X - \{\gamma\}$ is sequentially compact since X is sequentially compact and has uncountable cofinality. Hence, $(X - \{\gamma\}) \times (\omega+1)$ is sequentially compact as is its subspace $Y_0 = \{p \in Y : p$ is the limit of a sequence in $Y - \{p\}\}$. Note that $Y_0 \neq \varnothing$ since $\langle \omega, 0 \rangle \in Y_0$. (In fact, Y_0 is dense in the set of nonisolated points of Y.)

Let $x \in Y$ and $y \in Y_0$. By Note 3.4, xy is a limit of a sequence in $Y - \{xy\}$ since y is a limit of a sequence in $Y - \{y\}$. That is $xy \in Y_0$. Therefore, Y_0 is a sequentially compact subsemigroup of Y. So by Grant's result [9], Y_0 is a topological group.

Let e be the identity element of Y_0. Since Y is cancellative, e is the identity of Y. Then by the last paragraph, $xe = x \in Y_0$ for all $x \in X - Y_0$. This is a contradiction since $\langle \gamma, 0 \rangle \in X - Y_0 \neq \varnothing$. Therefore, Y does not admit a cancellative topological semigroup structure. ❑

Note that the proof of Theorem 3.10 also goes through if each X_α were only sequentially compact. That is, this modified Y would be pseudocompact but not countably compact and would not admit a cancellative topological semigroup structure. However, this modified Y would not be locally compact if some X_α were not.

QUESTION 3.11: In the view of the Tychonoff plank type space considered in Theorem 3.10, it is natural to ask the following question:

Let S be a sequentially compact space and $p \in S$ be the limit of a nontrivial sequence in X. Does the pseudocompact, not countably compact space $Y^1 = (X \times S) - \{\langle \gamma, p \rangle\}$ admit a cancellative topological semigroup structure, where X is a long-line like space using an ordinal γ of uncountable cofinality and compact first countable spaces X_α for each $\alpha \in \gamma$?

Note that the proof of Theorem 3.10 breaks down if there are nontrivial convergent sequences in $\{\gamma\} \times (S - \{p\})$.

We can solve this problem in certain cases. For example let $Y' = ((\omega_1 + 1) \times [0,1]) - \{(\omega_1, 1)\}$. Then Y' does not admit a cancellative topological semigroup structure by Proposition 3.7 (where the partition of Y' for 3.7 is $\{(\omega_1 \times [0,1], \{\omega_1\} \times [0,1)\})$.

More generally, if R is sequentially compact containing a copy of $\gamma + 1$, does $(R \times S) - \{\langle\gamma, p\rangle\}$ admit a cancellative topological semigroup structure? How about the case where $\gamma + 1$ is replaced by a cofinal subset containing γ and all smaller ordinals of countable cofinality?

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Factorization of Metrizability

H. H. HUNG

Department of Mathematics
Concordia University
Montréal, Québec, Canada H4B 1R6

ABSTRACT: A general method of factorization in introduced. It consists of, first, the weakening of the concept of an open base or that of a network, and, second, the restoration to the weakened concept its full vigor. Thus, every time, we have the restorative + the weakened concept = the original concept. The workings of the method are exceedingly simple. Its field of applications goes well beyond metrizability. Results include an improvement of Heath and Aull on the developability and metrizability of semistratifiable spaces. We also have that every Nagata space has a metrizable dense subset.

In spite of the title, our discourse here does not limit itself to the factorization of *metrizability*. Rather, it presents to the readers a method that is applicable to the factorization of a large number of properties, of which metrizability is but the most prominent example. The method of factorization consists of, first, the weakening of the concept of an open base or that of a network, and, second, the restoration to the weakened concept its full vigor. Thus, every time, we have the restorative + the weakened concept = the original concept. The workings of the method are exceedingly simple and its applications encompass notably an improvement of Heath's celebrated factorization of developability [4] (see also [6] and [8]) and the metrization of stratifiable spaces (Remark 1 on Theorem 2.1 and Theorem 3.2).

0. NOTATIONS AND TERMINOLOGY

1: Given a collection $\mathcal{U}$ of subsets on a topological space X. We write $\bar{\mathcal{U}}$ for $\{ClU : U \in \mathcal{U}\}$. For every $x \in X$, we write $\mathcal{U}(x)$ for $\{U \in \mathcal{U} : x \in U\}$. If, given $x \in X$, for every neighborhood W of x, there is $U \in \mathcal{U}(x)$ such that $U \subset W$, we say $\mathcal{U}(x)$ is *basic*.

2: We write $[X]^{\leq\omega}$ for the family of all *countable* subsets of X, given a set X.

3: Given a nonisolated point x in a topological space X, we let $a(X,x) \equiv \min\{|A| : A \subset X\backslash\{x\}$ and $x \in ClA\}$. Further, $a(X) \equiv \sup\{a(X,x) : x$ is a nonisolated point in $X\}$, if X is nondiscrete.

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4: A topological space is *virtually* collectionwise normal if, given any discrete family C of closed sets, there is a σ-discrete family $\mathcal{U}$ of open sets so that every member of $\mathcal{U}$ intersects exactly one member of C and so that $\bigcup C \subset \bigcup \mathcal{U}$ (see (*) of Theorem 4 of [5]).

5: A topological space X is *virtually developable* if there is a σ-discrete family $C = \bigcup\{C_n : n \in \omega\}$ of closed subsets, to each member C of which is associated a countable family $\mathcal{U}(C)$ of open subsets in such a manner that, for every $\xi \in X$, $\{U : \xi \in U \in \mathcal{U}(C), C \in C(\xi)\}$ is a local base at ξ (see Theorem 1 and Remarks on Theorem 4 of [5]). If, in particular, for each C, $\mathcal{U}(C)$ is a singleton and its sole member contains C, we have a developable space.

6: A topological space X is a *σ-space* if there is a σ-discrete family $C = \bigcup\{C_n : n \in \omega\}$ of subsets such that, for every $\xi \in X$, $C(\xi)$ is *basic* (see 1 above).

7: For definitions of *semistratifiable* and *stratifiable spaces*, we adopt the characterization as laid out in Section 5.8 of [2], keeping in mind the convention that stratifiable spaces are always T_1 (see, e.g., [10, Definition VI.9]).

8: Given a topological space X. A collection $\mathcal{U}$ of subsets on X is said to be a *base for selected accumulations* (BSA) if, on it, there is a function $f : \mathcal{U} \to [X]^{\leq\omega}$, such that, for every $\xi \in X$, $\xi \in Cl \bigcup f[\mathcal{U}(\xi)]$. For convenience, we refer to $f(U)$ as the *adumbration* of U, for any $U \in \mathcal{U}$, and $\bigcup f[\mathcal{U}(\xi)]$ as the *adumbration* of $\mathcal{U}(\xi)$. Examples of BSA's are networks (and of course open bases). Indeed, a network $\mathcal{U}$ is such a BSA that the adumbration of every $U \in \mathcal{U}$ can be taken to be within U, in which case we say f is *shrinking*, and of cardinality 1 (in which case, we say f is *thin*). In particular, separable spaces have open BSA's of cardinality 1, and any topological space that has a BSA of cardinality 1 is separable (Proposition 1.1). Semistratifiable spaces have σ-discrete closed BSA's (Proposition 1.2 below).

9: Given a topological space $(X, \mathcal{T})$. A function $A : X \to 2^X$ is called an *antecedent*, if, for every $\xi \in X$, $\xi \in Cl A(\xi)$. We write X_A for $\{\xi \in X : \xi \notin A(\xi)\}$. If, given any antecedent A on X, there is a function $H_A : \bigcup A\,[X_A] \times \omega \to \mathcal{T}$ such that:

(i) $x \notin H_A(x,n)$, for any $x \in \bigcup A[X_A]$, $n \in \omega$ (Altruism),

(ii) for all $\xi \in X_A$, $H_A[A(\xi) \times \omega](\xi)$ is a local base at ξ (Neighborliness),

we say X is an *H_*-space.*

(Note that it is only required that an open collection, large enough to contain local bases at all points of X_A, be *indexed* by the set $\bigcup A[X_A] \times \omega$ in a certain manner. Whether it is point-countable is clearly immaterial, even though for a T_1-space with a point-countable base the required indexing is obvious.)

If the range of H_A is relaxed to 2^X and $H_A[A(\xi) \times \omega](\xi)$ is required only to be *basic* (Section 0.1), we say X is a *sub-H_**-space.

The notion of an H_*-space, it must be explained, evolved from that of an H-space [8]. While the former is sufficient for our purpose (Theorems 2.1, 2.2, 3.1, and 3.2), only the latter is strong enough to force a β-space to have a BCO (cf. Lemma 3.2 of [3] and [11]).

1. ELEMENTARY RESULTS

PROPOSITION 1.1: Separable spaces are spaces with BSA's of cardinality 1, and conversely.

Proof: Given a topological space X with a countable dense subset A. $\{X\}$ is a BSA with an f so defined that $f(X) = A$. Conversely, if X is a topological space with a BSA of cardinality 1, the lone element of the BSA has to be X and the *adumbration* (countable) of X is dense in it. ❑

PROPOSITION 1.2: On any semistratifiable space X, there is a σ-discrete closed BSA $\mathcal{U}$. Given a closed subset C on X, we can assume the *adumbration* of $\mathcal{U}(\xi)$ to be within C, if $\xi \in C$.

Proof: Given X as described in Theorem 5.8 of [2]. Let the points of X be well ordered by <. For any $x \in X$, $m, n \in \omega$, let

$$U(x, m, n) \equiv g(m, x)\backslash\bigcup\{g(m, y) : y < x\}\backslash\bigcup\{g(n, z) : z \notin g(m, x)\}.$$

Clearly, for every $m, n \in \omega$, $\mathcal{U}_{m,n} \equiv \{U(x, m, n) : x \in X\}$ is discrete and $\mathcal{U} \equiv \bigcup\{\mathcal{U}_{m,n} : m, n \in \omega\}$ is a σ-discrete closed BSA, if f is so defined that $f(U(x, m, n)) = \{x\}$ for all $x \in X$, $m, n \in \omega$. The second assertion follows, if one replaces, for all $m \in \omega$, $g(m,x)$ by $g(m,x)\backslash C$ whenever $x \notin C$. ❑

REMARK: It follows from the second assertion that, for any countable closed cover $C = \{C_0, C_1, C_2, \ldots\}$ of X, there are σ-discrete closed BSA's $\mathcal{U}^{(0)}$, $\mathcal{U}^{(1)}$, $\mathcal{U}^{(2)}$, ..., with $f^{(0)}, f^{(1)}, f^{(2)}$, ..., such that, for every $n \in \omega$, $\bigcup f^{(n)}[\mathcal{U}^{(n)}(\xi)] \subset C_n$, if $\xi \in C_n$.

PROPOSITION 1.3: On any stratifiable space X, there is a σ-discrete open BSA $\mathcal{U}$, such that, for every $\xi \in X$, $\bigcap(\mathcal{U}(\xi))^- = \{\xi\}$.

PROPOSITION 1.4 (Hung [5]): Regular T_0-spaces, *virtually* collectionwise normal and *virtually* developable, are metrizable.

2. PRIMARY RESULTS

THEOREM 2.1: On an H_*-space X with $a(X) = \omega$, σ-HCP (respectively, σ-discrete, σ-locally finite, locally countable, σ-disjoint, σ-point-finite, point-countable) open BSA's $\mathcal{U}$ beget σ-HCP (respectively, σ-discrete, σ-locally finite, locally countable, σ-disjoint, σ-point-finite, point-countable) open bases.

Proof: We note that X is first countable and therefore, for every $x \in X$, there is a local base $\{V(x,n) : n \in \omega\}$. For each $\xi \in X$, let $A(\xi)$ be $\bigcup f[\mathcal{U}(\xi)]$. Clearly,

$$\{U \cap H_A(x, n), U \cap V(x, n) : x \in f(U), n \in \omega,\ U \in \mathcal{U}\}$$

is a σ-HCP (respectively, σ-discrete, σ-locally finite, locally countable, σ-disjoint, σ-point-finite, point-countable) open base. ❑

REMARK 1: Clearly (from the proof), if f is *shrinking* (Section 0.8), we have σ-HCP (respectively, σ-discrete, σ-locally finite, locally countable, σ-disjoint,

σ-point-finite, point-countable) open bases on some *dense* subsets of X, *regardless* whether X is an H_*-space, as long as it is first countable. In particular, Nagata Spaces, being σ-spaces, have σ-networks C on which are *shrinking* f's and therefore (we can immediately see) *dense* metrizable subspaces (if we note their first countability and their collectionwise normality). Among H_*-spaces, Nagata spaces are of course metrizable (see also Theorem 3.2).

REMARK 2: If $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in w\}$ is a σ-HCP open *base* on a T_3-space X, then $\mathcal{U}_n$, for every $n \in \omega$, is point-finite at points that are not isolated. On the other hand, if $\mathcal{U}$ is only a BSA, we cannot draw the same conclusion.

REMARK 3: On sub-H_*-spaces, clearly we have an analogue of Theorem 2.1 where BSA's beget networks.

REMARK 4: When Theorem 2.1 is applied to point-countable open BSA's we have a result in an area addressed in Lemma 2 of [9], where there is, however, so much overlap that the authors *conjectured* one of the factors superfluous.

THEOREM 2.2: Given a topological space X with a σ-discrete closed BSA C and with $a(X) = \omega$. X is *virtually developable* if it is an H_*- space.

Proof: We note again the first countability of X and a local base $\{V(x,n): n \in \omega\}$ at every $x \in X$. For each $\xi \in X$, let $A(\xi) = \bigcup f[C(\xi)]$. Clearly, for every $C \in C$, the family

$$\{H_A(x, n), V(x, n): x \in f(C), \ \ n \in \omega\}$$

is countable and can be considered the $\mathcal{U}(C)$ in the Definition of virtual developability (Section 0.5). ❑

REMARK: Theorem 2.2 above in conjunction with Proposition 1.4 factors metrizability into many very weak properties.

3. DEVELOPABILITY OF SEMISTRATIFIABLE SPACES AND METRIZABILITY OF STRATIFIABLE SPACES

To address the problem of developability of semistratifiable spaces, we are to strengthen Theorem 2.2. We are to weaken the concept of an H_*-space (Section 0.9) by placing a restriction on the *antecedent* A in the form of a countable closed cover C, requiring, for every $\xi \in X$, that $\xi \in Cl(A(\xi) \cap C)$ for every $C \in C(\xi)$. Such an A is said to be an antecedent *restricted to* C and such an X an H_*-space *restricted* by C. Spaces X with a $\delta\theta$-base $\mathcal{U} = \bigcup\{\mathcal{U}_n: n \in \omega\}$ such that, for each $\xi \in X$, $\bigcup\{\mathcal{U}_n(\xi) : |\mathcal{U}_n(\xi)| \le \omega\}$ is a local base at ξ, is of course an H_*-space *restricted* by

$$\{\{x \in X : |\mathcal{U}_n(x)| \le \omega\} : n \in \omega\}.$$

THEOREM 3.1: A semistratifiable H_*-space X restricted by any countable closed cover C, with $a(X) = \omega$, is developable.

REMARK: Theorem 3.1 above clearly generalizes Aull [1].

To address the problem of metrizability of stratifiable spaces, we note Proposition 1.3 and see that the family $\{U \cap H_A(x,n) : x \in f(U), n \in \omega, U \in \mathcal{U}(\xi)\}$, as it appears in the proof of Theorem 2.1, constitutes a local base at $\xi \in X_A$, *if* we have

(ii′) $\bigcap(H_A[A(\xi) \times \omega](\xi))^-$ be countably compact and have nonvoid intersection with any closed Γ unless Γ is disjoint from some member of the family $(H_A[\mathrm{A}(\xi) \times \omega](\xi))^-$.

If (ii) in the Definition of H_*-spaces (Section 0.9) is replaced by (ii′) above, we say we have BH_*-spaces.[a] BH_*-spaces, like H_*-spaces, can, of course, be restricted by some C. We therefore have the following theorem.

THEOREM 3.2: A stratifiable BH_*-space X restricted by any countable closed cover C, with $a(X) = \omega$, is metrizable.

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[a]In [7], we factored a metrizing structure into two parts, A and B. The latter came into being with the kind of weakening that we are experiencing here when we go from (ii) to (ii′). Hence the notation.

Iwasawa-type Decomposition in Compact Groups

GERALD ITZKOWITZ[a,c] AND TA SUN WU[b]

[a]*Department of Mathematics*
Queens College
City University of New York
Flushing, New York 11367
and
[b]*Department of Mathematics*
Case Western Reserve University
Cleveland, Ohio 44106

ABSTRACT: Iwasawa in his famous paper "On some types of topological groups," proved a number of decomposition theorems for compact groups. Among them is the theorem stating that a compact connected group G can be written as $G = Z(G) \cdot [G, G]$, where $Z(G)$ is the center of G and $[G, G]$ is the commutator subgroup of G. Later authors have refined this result to the statement that $G = Z(G)_o \cdot [G, G]$, where $Z(G)_o$ is the component of the identity of $Z(G)$. In the present paper we show that these theorems may be further strenghthened to the following statement: Let G be a compact group and let N be a compact normal subgroup of G. If either $G = A \cdot N$, where A is a compact connected Abelian group, or if G/N is connected, then $G = Z_G(N)_o \cdot N$, where $Z_G(N)_o$ is the component of the identity of the centralizer in G of N. The proof of this result has a number of interesting corollaries.

1. INTRODUCTION

In his famous paper [7], Iwasawa proved a number of decomposition theorems. Among them is the theorem: "Let G be a connected topological group and N be a compact normal subgroup of G. If $N_1 = [N, N]$ is the commutator subgroup of N and if $Z(N)$ is the center of N, then $N = N_1 \cdot Z(N)$ and $N_1 \cap Z(N)$ is a totally disconnected group." As noted in his paper if $G = N$ then we have a nice structure decomposition theorem for G, namely that each compact connected group G can be written in the form $G = Z(G) \cdot G_1$, where G_1 is the commutator subgroup of G and $G_1 \cap Z(G)$ is totally disconnected. Other authors (see Moskowitz [8]) have extended this result to show that every reductive compact Lie group has the decomposition $G = Z(G)_o \cdot G_1$, where $Z(G)_o$ is the component of the identity of $Z(G)$. Furthermore, in his paper, Moskowitz even extends this result to include all pro-reductive groups(projective limits of reductive groups) and in particular to com-

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pact connected groups. It is the purpose of this paper to show that Iwasawa's and Moskowitz's results are special cases of the following theorem.

THEOREM 7: Let G be a compact group and let N be a compact normal subgroup of G. If either $G = A \cdot N$ where A is a compact connected Abelian group or if G/N is connected then:

(1) $G \cong Z_G(N)_o \cdot N$, where $Z_G(N)_o$ is the component of the identity of the centralizer of N in G;

(2) G contains a subgroup $J \subset Z(G) \cap N$ such that $G/J \cong Z_G(N)_o/J \times N/J$.

We will be making use of the following Theorems A and B that appear in [2, III: Section 7.3]. A family of normal subgoups $\{H_\alpha: \alpha \in I\}$ of the topological group G, where I is a directed set, satisfies the [AP] condition if

(1) $H_\alpha \supset H_\beta$, whenever $\alpha \leq \beta$,

(2) each H_α, $\alpha \in I$, is closed in G, and

(3) each neighborhood of the identity contains one of the H_α.

THEOREM A: If G is a T_2 topological group with a family of normal subgroups $\{H_\alpha: \alpha \in I\}$ that satisfies the [AP] condition and if each H_α is compact or complete then G is the projective limit $G = \operatorname{proj} G/H_\alpha$ of the quotient groups G/H_α.

Let $G_\alpha = G/H_\alpha$. Note that associated with the projective limit is the collection of surjective homomorphisms $f_{\alpha\beta}: G_\beta \to G_\alpha$, satisfying $f_{\alpha\beta}\pi_\beta = \pi_\alpha$ for $\alpha \leq \beta$, where for each γ, $\pi_\gamma: G \to G_\gamma$ is the restriction of the projection map $\pi_\gamma: \Pi_{\alpha \in I} G_\alpha \to G_\gamma$.

THEOREM B: Suppose that G is T_2 and that the H_α, $\alpha \in I$, are compact and satisfy the [AP] condition.

(a) Let L be a closed subgroup of G; then for each $\alpha \in I$, the subgroup $L_\alpha = \pi_\alpha(L) \subset G_\alpha$ is closed and the isomorphism $\psi: G \to \operatorname{proj} G_\alpha$ gives by restriction an isomorphism of L onto $\operatorname{proj} L_\alpha$. If L is also normal in G, then L_α is normal in G_α, for each $\alpha \in I$ and ψ induces an isomorphism of G/L onto $\operatorname{proj} G_\alpha/L_\alpha$.

(b) Conversely, for each $\alpha \in I$ let L_α be a closed subgroup of G_α such that $L_\alpha = f_{\alpha\beta}(L_\beta)$, whenever $\alpha \leq \beta$. Then there is a unique closed subgroup L of G such that $L_\alpha = \pi_\alpha(L)$ for each $\alpha \in I$, and if in addition L_α is normal in G, for each $\alpha \in I$, then L is normal in G.

2. CLASSICAL-STYLE DECOMPOSITIONS

In this section we consider some classical-type decompositions of compact connected Lie groups and show how the cited theorems on projective limits extend these to compact connected groups. A corollary of Theorem B is the following:

THEOREM 1: Let G be a compact group and suppose the collection $\{H_\alpha: \alpha \in I\}$ of normal subgroups satisfy the [AP] condition, so that $G = \operatorname{proj} G_\alpha$, where each $G_\alpha = G/H_\alpha$. Suppose that for each $\alpha \in I$, $G_\alpha = K_\alpha \cdot N_\alpha$, where K_α and N_α are closed subgroups and N_α is normal in G_α. If the inverse system of surjective maps $f_{\alpha\beta}: G_\beta \to G_\alpha$, $\alpha \leq \beta$, satisfy $f_{\alpha\beta}(N_\beta) = N_\alpha$ and $f_{\alpha\beta}(K_\beta) = K_\alpha$, then:

(a) $K = \text{proj}\, K_\alpha$ is a closed subgroup of G,
(b) $N = \text{proj}\, N_\alpha$ is a closed normal subgroup of G, and
(c) $G = K \cdot N$.

Proof: (a) and (b) are immediate from Theorem B. Since $\pi_\alpha : G \to G_\alpha$ is a homomorphism $\pi_\alpha(K \cdot N) = \pi_\alpha(K) \cdot \pi_\alpha(N) = K_\alpha \cdot N_\alpha = G_\alpha$, for each α. Since K and N are closed in G they are compact and so $K \cdot N$ is a compact and therefore closed subgroup of G. By the uniqueness of the closed subgroup $K \cdot N$ satisfying $\pi_\alpha(K \cdot N) = K_\alpha \cdot N_\alpha = G_\alpha$, it follows that $G = K \cdot N$. ❑

DEFINITIONS AND NOTATION: Let G be a topological group. Then:
(1) $Z(G) = \{x \in G : xy = yx \text{ for all } y \in G\}$ is the center of G.
(2) G_o is the component of the identity in G.
(3) $G_1 = [G, G]$ is the closure in G of the subgroup generated by the elements $xyx^{-1}y^{-1}$, where $x,y \in G$. G_1 is called the commutator subgroup of G.
(4) $Z_G(N) = \{x \in G : xy = yx \text{ for all } y \in N\}$, where N is a subgroup of G. $Z_G(N)$ is called the centralizer of N in G.
(5) $N_G(T) = \{y \in G:\ yTy^{-1} = T\}$ where T is an Abelian subgroup of G. $N_G(T)$ is called the normalizer of T in G.
(6) If G is a compact group, a maximal protorus is a maximal connected Abelian subgroup of G. It is known (see [1]) that if $G = \text{proj}\, G_\alpha$, where each G_α is a compact connected Lie group then each maximal protorus $T = \text{proj}\, T_\alpha$, where each T_α is a maximal torus in G_α. A torus T in a compact Lie group G is a group of the form R^n/Z^n where R is the real numbers and Z is the integers.

It is an elementary fact (see [3, 6.14]) that $\text{proj}\, G_\alpha$ is a closed subgroup of ΠG_α when the G_α are T_o and it is a folklore theorem that a projective limit of compact connected groups is compact connected. For completeness we give a proof: If not, let $G = \text{proj}\, G_\alpha$, where each G_α is compact connected. (It is clear that G is compact if each G_α is compact.) Since G is not connected, it contains a proper open and closed neighborhood of the identity which by [3, 7.6] contains an open and closed proper normal subgroup N. Clearly, each $\pi_\alpha(N) = N_\alpha$ is an open and closed subgroup of G_α so that $\pi_\alpha(N) = G_\alpha$. Since $\pi_\alpha(G) = G_\alpha$, for each α, Theorem B implies $G = N$, a contradiction.

It is also a classical result [4] that for a compact connected Lie group G we have $G = Z(G)_o \cdot G_1$. Consequently, since $Z(G)_o \subset T$, where T is a maximal protorus in G, it follows that $G = T \cdot G_1$. Furthermore, it is well known that G_1 is normal in G.

EXAMPLE 1: If G is a compact connected group then it is a classical result due to Yamabe [10] that $G = \text{proj}\, G_\alpha$, where each G_α is a compact connected Lie group. As noted previously the maximal protorus T in G satisfies $T = \text{proj}\, T_\alpha$, where each T_α is a maximal torus in G_α. Furthermore, the restriction of the projection π_α to G_1 is a surjection onto $G_{\alpha,1} = [G_\alpha, G_\alpha]$ so that $G_1 = \text{proj}\, G_{\alpha,1}$. Therefore, by Theorem 1, $G = T \cdot G_1$.

EXAMPLE 2. Moskowitz [8] shows that if G is a compact connected Lie group and if $\psi: G \to J$ is a continuous epimorphism, then $\psi(Z(G)_o) = Z(J)_o$. Thus, in the case where G is a compact connected group, if $G = \operatorname{proj} G_\alpha$, where each G_α is a compact connected Lie group, then the associated homomorphisms $f_{\alpha\beta}: G_\beta \to G_\alpha$, $\alpha \le \beta$, are open continuous surjective maps. Therefore, $f_{\alpha\beta}(Z(G_\beta)_o) = Z(G_\alpha)_o$. Thus, Theorem B tells us that there is a unique closed group L in G such that $\pi_\alpha(L) = Z(G_\alpha)_o$ for each α, and $L = \operatorname{proj} Z(G_\alpha)_o$. Since $Z(G)_o$ is connected and each π_α is continuous it is clear that $\pi_\alpha(Z(G)_o) \subset Z(G_\alpha)_o$. Furthermore, as noted already the projective limit of compact connected groups is compact connected so that $L \subset Z(G)_o$. Therefore, $\pi_\alpha(L) \subset \pi_\alpha(Z(G)_o) \subset Z(G_\alpha)_o = \pi_\alpha(L)$, so that $\pi_\alpha(Z(G_\alpha)_o) = Z(G_\alpha)_o$ for each α. Thus by uniqueness $Z(G)_o = \operatorname{proj} Z(G_\alpha)_o$. (This is a quick proof of [8, 2.3]). Since $G_1 = \operatorname{proj} G_{\alpha,1}$ and since $G_\alpha = Z(G_\alpha)_o \cdot G_{\alpha,1}$, it follows that $G = Z(G)_o . G_1$.

NOTE: Example 1 shows that all compact connected G satisfy $G = A \cdot N$, where $A = T$ is a compact connected Abelian subgroup and $N = G_1$ is a compact normal subgroup. Example 2 shows that all such G satisfy $G = Z(G)_o \cdot N$, where $N = G_1$, and this is a special case of our Theorems 4 and 7.

3. INNER AUTOMORPHISMS OF COMPACT NORMAL SUBGROUPS

Let G be a topological group.

DEFINITION 1: $\operatorname{Aut} G$, the set of automorphisms of G, consists of all topological isomorphisms of G onto itself.

DEFINITION 2: $\operatorname{Inn} G = \{\tau \in \operatorname{Aut} G: \tau(x) = yxy^{-1}$, for some $y \in G\}$ is the set of inner automorphisms of G.

We will be making use of the following famous theorems of Iwasawa [7, Theorems 1and 4].

THEOREM C: If G is a compact group then $\operatorname{Aut} G/\operatorname{Inn} G$ is a totally disconnected group.

THEOREM D: A compact Abelian normal subgroup of a connected topological group G is contained in the center of G.

An immediate consequence of Iwasawa's theorem is the following lemma.

LEMMA 1: Let G be a compact group and let $G = A \cdot N$, where A is a compact connected Abelian subgroup and N is a compact normal subgroup of G. Let $\pi: A \to \operatorname{Aut} N$ be defined by $\pi(x)(y) = xyx^{-1}$, $y \in N$. Then $\pi(A) \subset \operatorname{Inn} N$.

Proof: By Theorem C the set $\operatorname{Aut} N/\operatorname{Inn} N$ is totally disconnected. Since A is connected $\pi(A)$ and therefore, $\pi(A)/\operatorname{Inn} N$ is connected. Therefore, $\pi(A)/\operatorname{Inn} N \subset (\operatorname{Aut} N/\operatorname{Inn} N)_o$ and so $\pi(A) \subset \operatorname{Inn} N$. ❑

The next lemma is folklore and it's proof is a modification of the proof of [7, Lemma 2.2].

LEMMA 2: If G is a compact connected group, if $H \subset Z(G)$, and if G/H is Abelian then G is Abelian.

Proof: Let $u \in G$ and let M be the group generated by u and H. Note that M is Abelian since u commutes with any power of itself and with any element of H. Let $\psi: G \to G/H$ be the natural map. Then if $m \in M$ and $x \in G$ we have $\psi(xmx^{-1}) = \psi(x)\psi(m)\psi(x)^{-1} = \psi(m)$ since G/H is Abelian. Therefore, $xmx^{-1} = mh \in MH = M$, and so M is normal. Therefore, by [7, Theorem 4], $M \subset Z(G)$. Since $u \in G$ was arbitrary G is Abelian. ❑

LEMMA 3: Let G be a compact group and let $G = A \cdot N$ where A is a compact connected subgroup and N is a compact normal subgroup of G. Then for each $a \in A$ there is an element $n_a \in N$ such that $n_a a \in Z(N)$ (that is, $n_a a$ centralizes N). Furthermore, $N_A = \{n_a \in N: an_a = n_a a \in Z(N),\ a \in A\}$ is a subgroup of N.

Proof: Let $\pi: A \to \operatorname{Aut} N$ be defined by $\pi(x)(y) = xyx^{-1}$, for $y \in N$. Then by Lemma 1, $\pi(A) \subset \operatorname{Inn} N$. Thus let $a \in A$ and $n_1 \in N$ be arbitrary. Then $\pi(a)(n_1) = an_1a^{-1} \in N$. Since $\pi(a) \in \operatorname{Inn} N$, there is $n \in N$ such that $\pi(a)(n_1) = nn_1n^{-1}$ for all $n_1 \in N$. Thus $an_1a^{-1} = nn_1n^{-1}$ or $(n^{-1}a)n_1(n^{-1}a)^{-1} = n_1$ for all $n_1 \in N$. If we let $n_a = n^{-1}$ we see that $n_a a \in Z(N)$. Thus $a = an_a n_a^{-1} = n_a^{-1} an_a$ so that $n_a a = an_a$.

Now let $n_1, n_2 \in N_A$. Then there are $a_1, a_2 \in A$ such that $a_1n_1, a_2n_2 \in Z(N)$. Also, if $an_a \in Z(N)$ then $(an_a)^{-1} = n_a^{-1}a^{-1} \in Z(N)$. Since $Z(N)$ is a group, multiplication in $Z(N)$ is closed. Thus, we have $a_2^{-1}a_1n_1n_2^{-1} = a_2^{-1}n_2^{-1}a_1n_1 = (n_2a_2)^{-1}a_1n_1 \in Z(N)$. Therefore, $a_2^{-1}a_1 = a_1a_2^{-1} \in A$ satisfies $(a_2^{-1}a_1)(n_1n_2^{-1}) \in Z(N)$ so that $n_1n_2^{-1} \in N_A$. ❑

LEMMA 4: Let G be a compact group and let N be a compact normal subgroup such that G/N is connected. Let $\theta: G \to \operatorname{Aut} N$ be defined by $\theta(x)(y) = xyx^{-1}$, $y \in N$. Then $\theta(G) \subset \operatorname{Inn} N$.

Proof: Since $\theta(N) \subset \operatorname{Inn} N$ we have $N \subset \theta^{-1}(\theta(N)) \subset \theta^{-1}(\operatorname{Inn} N)$.

By [3, 5.35],

$$G/\theta^{-1}(\operatorname{Inn} N) \cong (G/N)/(\theta^{-1}(\operatorname{Inn} N)/N),$$

so that there is a continuous homomorphism $\tau: G/N \to G/\theta^{-1}(\operatorname{Inn} N)$. By [3, 5.34],

$$G/\theta^{-1}(\operatorname{Inn} N) \cong \theta(G)/\operatorname{Inn} N \subset \operatorname{Aut} N/\operatorname{Inn} N.$$

This means that there is a continuous homomorphism of G/N into $\operatorname{Aut} N/\operatorname{Inn} N$. But G/N is connected and so $\theta(G)/\operatorname{Inn} N$ is connected. Since $\operatorname{Aut} N/\operatorname{Inn} N$ is totally disconnected it follows that $\theta(G)/\operatorname{Inn} N = \{e_{\operatorname{Aut} N/\operatorname{Inn} N}\}$ so that $\theta(G) \subset \operatorname{Inn} N$. ❑

THEOREM 2: Let G be a compact group and let N be a closed normal subgroup of G such that G/N is connected. Then $G = Z_G(N) \cdot N$

Proof: By Lemma 4, the map $\theta: G \to \operatorname{Aut} N$, defined by $\theta(x)(y) = xyx^{-1}$, $y \in N$, is into $\operatorname{Inn} N$. Thus if $x \in G$, $\theta(x) \in \operatorname{Inn} N$, so there is $n \in N$ such that $xyx^{-1} = \theta(x)(y) = n^{-1}yn$, for all $y \in N$. This means that $(nx)y(nx)^{-1} = y$, for all $y \in N$. Thus, $nx \in Z_G(N)$. But then for each $x \in G$, we have $x = n^{-1}nx = nxn^{-1} \in Z_G(N) \cdot N$, proving the theorem. ❑

NOTE: If N is a normal subgroup of G, then $Z_G(N)$ is also a subgroup. To see this, let $x,y \in Z_G(N)$. Then $yn = ny$, for all $n \in N$, and therefore, $ny^{-1} = y^{-1}n$, for all $n \in N$, so that $y^{-1} \in Z_G(N)$. However, this means that $xy^{-1}n = xny^{-1} = nxy^{-1}$, for all $n \in N$, so that $xy^{-1} \in Z_G(N)$. In the case where G is a compact subgroup and N is a compact normal subgroup of G, the subgroup $Z_G(N)$ and hence $Z_G(N)_o$ are compact subgroups of G.

4. $Z(G)_o, Z_G(N)_o$ AND THE STRUCTURE OF COMPACT GROUPS

In this section we show that the component of the identity of the center of a compact group G and of the centralizer of a compact normal subgroup can play a central role in the structure of the group. Specifically, we will show that if N is a compact normal subgroup of G and if either $G = A \cdot N$, where A is a compact connected Abelian group, or if G/N is connected, then $G = Z_G(N)_o \cdot N$. We show that in the first case, where A is Abelian that $Z(G)_o = Z_G(N)_o$ so that $G = Z(G)_o \cdot N$. These facts lead to some interesting corollaries about automorphism groups, normalizer subgroups of maximal protori of compact connected groups, and commutator subgroups of compact groups.

THEOREM 3: Let G be a compact group satisfying $G = A \cdot N$, where A is a compact connected Abelian subgroup and N is a compact normal subgroup of G. Then $G = Z_G(N)_o \cdot N$.

Proof: We first show that $G = Z_G(N) \cdot N$. Let $x \in G$ so that $x = an$, where $a \in A$ and $n \in N$. However, by Lemma 3, there is $n_a \in N$ such that $n_a a = an_a \in Z(N)$. Thus we can write

$$x = an = an_a n_a^{-1} n = (n_a a)(n_a^{-1} n) \in Z_G(N) \cdot N.$$

Therefore, $G \subset Z_G(N) \cdot N \subset G$, so that $G = Z_G(N) \cdot N$.

Now let $\psi\colon G \to G/N$ be the natural map so that $G/N \cong AN/N = \psi(A)$. Since A is connected and ψ is continuous it follows that G/N is connected. By [3,5.29] ψ is open and by [3, 7.3], $\psi(Z_G(N)_o) = \psi(Z_G(N))_o$. By [3, 7.3], $Z_G(N)/Z_G(N)_o$ is totally disconnected. Now $\psi(G) = G/N = (Z_G(N) \cdot N)/N = \psi(Z_G(N))$, so that $\psi(Z_G(N))$ is connected. However, since $\psi(Z_G(N)_o)$ is the component of the identity in $\psi(Z_G(N))$ it follows that $\psi(Z_G(N)_o) = \psi(Z_G(N)) = G/N$. Therefore, $G = Z_G(N)_o \cdot N$. ❑

THEOREM 4: Let G be a compact group satisfying $G = A \cdot N$, where A is a compact connected Abelian subgroup and N is a compact normal subgroup of G. Then $G = Z(G)_o \cdot N$.

Proof: By Theorem 3, $G = Z_G(N)_o \cdot N$. We will show that $Z_G(N)_o$ is central in G. To do this we first show that $Z_G(N)_o$ is Abelian. With ψ as in Theorem 3, we have seen that $\psi(A) = G/N$, and since A is Abelian so is $\psi(A)$. However, as noted in Theorem 3, $G/N = \psi(Z_G(N))$ so that $\psi(Z_G(N))$ and also $\psi(Z_G(N)_o) = \psi(Z_G(N))_o$ is Abelian. However, by [3, 5.33]

$$\psi(Z_G(N)_o) \cong (Z_G(N)_o \cdot N)/N \cong Z_G(N)_o/Z_G(N)_o \cap N$$

so that $Z_G(N)_o/Z_G(N)_o \cap N$ is Abelian. Also, $Z_G(N)_o \cap N$ is normal and Abelian in $Z_G(N)_o$. Therefore, $Z_G(N)_o \cap N$ is in the center of $Z_G(N)_o$ by Theorem D. By Lemma 2, it follows that $Z_G(N)_o$ is Abelian.

Since $G = Z_G(N)_o \cdot N$, if $x \in G$, we can write $x = an$, where $a \in Z_G(N)_o$ and $n \in N$. Since $Z_G(N)_o$ is Abelian, if we take $b \in Z_G(N)_o$ then $bx = ban = abn = anb = xb$, so that $Z_G(N)_o$ is central in G, that is $Z_G(N)_o \subset Z(G)$. Since $Z_G(N)_o$ is connected, $Z_G(N)_o \subset Z(G)_o$ and so $G = Z_G(N)_o \cdot N \subset Z(G)_o \cdot N \subset G$. Thus $G = Z(G)_o \cdot N$. ❑

NOTE: Since $Z(G) \subset Z_G(N)$, for any proper subgroup N of G, it follows that $Z(G)_o \subset Z_G(N)_o$. Thus, from the proof of Theorem 4 it follows that if G is compact and if $G = A \cdot N$ where A is a compact connected Abelian subgroup and N is a normal compact subgroup then $Z(G)_o = Z_G(N)_o$.

THEOREM 5: Let G be a compact group satisfying $G = T \cdot N$, where T is a maximal protorus of G and N is a compact normal subgroup of G. Let Φ be an automorphism of G. Then there exists an element $n \in N$ such that $\Phi(T) = n^{-1}Tn$.

Proof: Since Φ is an automorphism $\Phi(T)$ is a maximal protorus of G. By [3, 7.12], $\Phi(G_o) = \Phi(G)_o = G_o$. Since G_o is the maximal connected subgroup of G, $\Phi(T) \subset G_o$. By Mycielski [9] (see also [6]), there is $x \in G_o$ such that $x\Phi(T)x^{-1} = T$. By Theorem 4 , $G = Z(G)_o \cdot N$ so that $x = zn$, where $z \in Z(G)_o$ and $n \in N$. Therefore, $zn\Phi(T)n^{-1}z^{-1} = T$, and so $n\Phi(T)n^{-1} = z^{-1}Tz = T$. Thus $\Phi(T) = n^{-1}Tn$, proving the theorem. ❑

As a corollary we can now derive a quick proof of Theorem 4.7 in [1]. It is shown in [5] that if G is a compact connected group , if η is a continuous surjective homomorphism of G onto H, and if T is a maximal protorus of G, then $\eta(T)$ is a maximal protorus in H.

COROLLARY 1: Let η: $G \to H$ be a continuous surjective homomorphism of the compact connected group G onto H. Let T be a maximal protorus in G, then $\eta(N_G(T)) = N_H(\eta(T))$.

Proof: Let $y \in N_H(\eta(T))$ and let $F = \eta^{-1}(\eta(T))$. Then $F = T \cdot N$ where N is the kernel of η. Let $x \in G$ be such that $\eta(x) = y$. Since $y\eta(T)y^{-1} = \eta(T)$, it follows that $x\eta^{-1}(\eta(T))x^{-1} = \eta^{-1}(\eta(T))$ or $xFx^{-1} = F$. Thus x defines an automorphism of F. Since T is a maximal protorus in G it is also a maximal protorus of F. Thus by Theorem 5, there is $n \in N$ such that $\eta(T) = n^{-1}Tn$, or $xTx^{-1} = n^{-1}Tn$. Thus, $nxTx^{-1}n^{-1} = nxT(nx)^{-1} = T$. Since N is normal, $nx = xn_1$ where $n_1 \in N$, so that $xn_1 \in N_G(T)$. Since $\eta(xn_1) = \eta(x) = y$ it follows that η maps $N_G(T)$ onto $N_H(\eta(T))$. Since the inclusion $\eta(N_G(T)) \subset N_H(\eta(T))$ is clear, the corollary follows. ❑

COROLLARY 2: Let G be a compact group and suppose $Z(G)$ is totally disconnected. Then the only decomposition of G in the form $G = A \cdot N$, where A is a compact connected Abelian subgroup and N is a compact normal subgroup is the trivial one, where $N = G$.

Proof: If $G = A \cdot N$, then $G = Z(G)_o \cdot N$. Thus, if $Z(G)$ is totally disconnected, then $Z(G)_o = \{e\}$ and $N = G$. ❑

NOTE: This generalizes the classical theorem that says that a compact connected semisimple Lie group G satisfies $G = [G, G] = G_1$.

COROLLARY 3: If G is a compact group with a decomposition of the form $G = A \cdot N$, where A is a compact connected Abelian subgroup and N is a compact normal subgroup then $Z_G(N)_o = Z(G)_o$. Therefore, if $Z(G)$ is totally disconnected then $Z_G(N)_o = \{e\}$ and $Z(G) = Z_G(N)$.

THEOREM 6: Let G be a compact group and let N be a compact normal subgroup of G. If G/N is connected then $G = Z_G(N)_o \cdot N$.

Proof: Note first that $Z_G(N)_o \cdot N \subset Z_G(N) \cdot N = G$ and that $Z_G(N)/Z_G(N)_o$ is totally disconnected. Furthermore, $Z_G(N)_o$ is a closed subgroup of $(Z_G(N)_o \cdot N) \cap Z_G(N)$. Thus by [3, 5.34],

$$(Z_G(N)/Z_G(N)_o)\Big/[((Z_G(N)_o \cdot N) \cap Z_G(N))/Z_G(N)_o] \cong Z_G(N)\Big/(Z_G(N)_o \cdot N) \cap Z_G(N).$$

This means that $Z_G(N)/(Z_G(N)_o \cdot N) \cap Z_G(N)$ is a continuous image of $Z_G(N)/Z_G(N)_o$ under the open homomorphism

$$\rho: Z_G(N)/Z_G(N)_o \to (Z_G(N)/Z_G(N)_o)\Big/[((Z_G(N)_o \cdot N) \cap Z_G(N))/Z_G(N)_o]$$

and is, therefore, also, totally disconnected. Note that $G = Z_G(N) \cdot N = Z_G(N) \cdot (Z_G(N)_o \cdot N)$. Thus, if we take $A = Z_G(N)$ and $H = Z_G(N)_o \cdot N$ in [3, 5.33] we get the isomorphisms

$$G/Z_G(N)_o \cdot N \cong Z_G(N) \cdot N/Z_G(N)_o \cdot N \cong Z_G(N)/(Z_G(N)_o \cdot N) \cap Z_G(N).$$

However,

$$G/Z_G(N)_o \cdot N \cong G/N\Big/(Z_G(N)_o \cdot N/N)$$

and so $G/Z_G(N)_o \cdot N$ is a continuous homomorphic image of the connected group G/N and it is therefore connected. This means that

$$Z_G(N)\Big/(Z_G(N)_o \cdot N) \cap Z_G(N)$$

is simultaneously totally disconnected and connected and so must be trivial. Therefore $G = Z_G(N)_o \cdot N$. ❑

LEMMA 5: Let $G = K \cdot H$ where G is a compact group, and suppose K and H are closed normal subgroups of G. Then $J = K \cap H$ is a closed normal subgroup and $G/J \cong K/J \times H/J$.

Proof: It is clear that J is a closed normal subgroup of G. Under the natural homomorphism $\tau: G \to G/J$ we see that $G/J = \tau(G) = \tau(K) \cdot \tau(H)$. However, $\tau(K)$ and $\tau(H)$ are compact and $\tau(J) = \tau(K) \cap \tau(H) = \{e_{G/J}\}$. By [3, 6.12], $G/J \cong \tau(K) \times \tau(H) = K/J \times H/J$. ❑

COROLLARY 4: Let G be a compact group and let N be a closed normal subgroup of G such that G/N is connected. Let $J = Z_G(N)_o \cap N$. Then $G/J \cong Z_G(N)_o/J \times N/J$.

Proof: By the note after Theorem 2, $Z_G(N)_o$ is a closed subgroup of G, and by Theorem 6, $G = Z_G(N)_o \cdot N$. Let $x \in G$, then $x = an$, where $a \in Z_G(N)_o$ and $n \in N$. Let $y \in Z_G(N)_o$, then

$$xyx^{-1} = anyn^{-1}a^{-1} = ann^{-1}ya^{-1} = aya^{-1} \in Z_G(N)_o,$$

since $a,y \in Z_G(N)_o$ and $Z_G(N)_o$ is a group. Therefore, $Z_G(N)_o$ is normal, and Corollary 4 follows from Lemma 5. ❑

LEMMA 6: If the compact group G can be written in the form $G = Z_G(N)_o \cdot N$, where N is a compact normal subgroup, then the group $J = Z_G(N)_o \cap N \subset Z(G)$.

Proof: Let $x \in J$ and $y \in G$. Then $y = zn = nz$, where $z \in Z_G(N)_o$ and $n \in N$. Clearly,

$$xy = x(nz) = (xn)z = (nx)z = n(xz) = n(zx) = (nz)x = yx,$$

so that $x \in Z(G)$. ❑

The previous work can now be summarized in the following theorem.

THEOREM 7: Let G be a compact group and let N be a compact normal subgroup of G. If either $G = A \cdot N$, where A is a compact connected Abelian subgroup, or G/N is connected then:
(1) $G \cong Z_G(N)_o \cdot N$, and
(2) G contains a subgroup $J \subset Z(G) \cap N$ for which $G/J \cong Z_G(N)_o/J \times N/J$.

Proof: Statement (1) was proved in Theorems 4 and 6, and statement (2) follows from Corollary 4 and Lemma 6 if we take $J = Z_G(N)_o \cap N$. ❑

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Intersection Properties of Open Sets. II[e]

I. JUHÁSZ,[a,e,h] ZS. NAGY,[b,f] L. SOUKUP,[c,g] AND Z. SZENTMIKLÓSSY[d]

[a,b,c]*Mathematical Institute of the Hungarian Academy of Sciences*

and

[d]*ELTE TTK*

ABSTRACT: A topological space is called P_2 (P_3, $P_{<\omega}$) if and only if it does not contain two (three, finitely many) uncountable open sets with empty intersection. We show that (i) there are 0-dimensional $P_{<\omega}$ spaces of size 2^ω, (ii) there are compact $P_{<\omega}$ spaces of size ω_1, (iii) the existence of a Ψ-like examples for (ii) is independent of ZFC, (iv) it is consistent that 2^ω is as large as you wish but every first countable (and so every compact) P_2 space has cardinality $\leq \omega_1$.

1. INTRODUCTION

In this paper we continue the investigations started in [4]. There P_2 spaces, i.e., spaces having no two uncountable disjoint open subsets, were considered. We solve some problems which were left open in that paper and strengthen some of its results. First we introduce some strengthenings of notion P_2.

DEFINITION 1.1: A topological space X has property P_n, where n is a natural number, if it is Hausdorff and given open sets $U_0, U_1, \ldots, U_{n-1}$ with $\bigcap\{U_i: i<n\} = \varnothing$ we have $|U_i| \leq \omega$ for some $i<n$. We say X is $P_{<\omega}$ if and only if it is P_n for each $n<\omega$. The space X is called *strongly* P_2 provided the intersection of two uncountable open sets is uncountable.

Clearly P_m spaces are P_n for $n<m$ and strongly P_2 spaces have property $P_{<\omega}$.

In Section 2 we construct $P_{<\omega}$-spaces: a 0-dimensional one of size 2^ω and two locally compact, first countable examples of size ω_1. One of the ZFC constructions of locally compact $P_{<\omega}$ spaces is due to S. Shelah [9] and it is included here with his kind permission.

In Section 3 we will see why the construction of a compact P_3 space can be expected to be much harder than that of a P_2 one: a Ψ-like example (see the Definition 1.2 below), which worked in the P_2 case, cannot be constructed for the P_3 case in ZFC.

On the other hand we show (Corollary 3.10) that it is consistent that 2^ω is as large as you wish and there is a Ψ-like $P_{<\omega}$-space of size 2^ω.

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[e]E-mail: juhasz@math-inst.hu.

[f]E-mail: zsiga@math-inst.hu.

[g]E-mail: soukup@math-inst.hu. The third author was supported by DFG (grant Ko 490/7-1) and Magyar Tudományért Alapítvány.

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DEFINITION 1.2: (1) Given an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ we define the topological space $X[\mathcal{A}]$ as follows: its underlying set is $\omega \cup \mathcal{A}$, the elements of ω are isolated and the neighborhood base of $A \in \mathcal{A}$ in $X[\mathcal{A}]$ is $\mathcal{B}_A = \{A \cup \{A\} \setminus n : n \in \omega\}$.

(2) A topological space is *Ψ-like* if and only if it is homeomorphic to $X[\mathcal{A}]$ for some almost disjoint $\mathcal{A} \subset [\omega]^\omega$.

It is straightforward that Ψ-like spaces are locally compact, first countable and 0-dimensional.

In Section 4 we present the proof of a partition theorem below the continuum which was used in [4] to show the it is consistent that 2^ω is big but every first countable (and so every compact) P_2 space has cardinality $\leq \omega_2$.

Finally, in Section 5 we strengthen this result (and solve Problem 10 of [4]) by showing that it is consistent that 2^ω is as large as you wish but every first countable (and so every compact) P_2 space is of size $\leq \omega_1$.

We use standard topological notation and terminology throughout (cf. [6]).

2. ZFC CONSTRUCTIONS OF $P_{<\omega}$-SPACES

PROPOSITION 2.1: There are 0-dimensional $P_{<\omega}$ spaces of size 2^ω.

First proof of Proposition 2.1: We show that the space X from [4, Theorem 3] is in fact $P_{<\omega}$. To start with, we recall its definition. Fix an independent family $\mathcal{F} \subset [\omega]^\omega$ of size 2^ω. The underlying set of X will be $\omega \cup \mathcal{F}$, where the elements of ω are isolated. The neighborhood base of $F \in \mathcal{F}$ will be $\mathcal{B}_F = \{V_F(\mathcal{G}) : \mathcal{G} \in [\mathcal{F} \setminus \{F\}]^{<\omega}\}$, where $V_F(\mathcal{G}) = \{F\} \cup F \setminus \bigcup \mathcal{G}$. If $\bigcup \mathcal{F} = \omega$ then X will be 0-dimensional and T_2.

Let $U_0, \ldots, U_{n-1}$ be uncountable open subsets of X. For each $i < n$ we can find $\{F^i_\nu : \nu < \omega_1\} \subset \mathcal{F}$ and $\{\mathcal{G}^i_\nu : \nu < \omega_1\} \subset [\mathcal{F}]^{<\omega}$ such that $F^i_\nu \notin \mathcal{G}^i_\nu$, the F^i_ν are all distinct and $V_{F^i_\nu}(\mathcal{G}^i_\nu) \subset U_i$.

Define the set mapping $f : \omega_1 \to [\omega_1]^{<\omega}$; by the stipulation

$$f(\nu) = \{\mu : \exists j < n \ F^j_\mu \in \bigcup_{i<n} \mathcal{G}^i_\nu\}.$$

Since the F^j_μ are different we have $|f(\nu)| \leq n |\bigcup_{i<n} \mathcal{G}^i_\nu|$. By Hajnal's theorem on set mappings [3] there is an uncountable f-free subset I of ω_1. Let $\nu_0, \ldots, \nu_{n-1}$ be different elements of I. Then $\{F^j_{\nu_j} : j < n\} \cap \bigcup_{i<n} \mathcal{G}^i_{\nu_i} = \emptyset$, so

$$\emptyset \neq \bigcap_{j<n} F^j_{\nu_j} \setminus \bigcap(\bigcup_{i<n} \mathcal{G}^i_{\nu_i}) = \bigcap_{i<n} V_{F^i_{\nu_i}}(\mathcal{G}^i_{\nu_i}) \cap \omega \subset \bigcap_{i<n} U_i,$$

which completes the proof. ❑

Second proof of Proposition 2.1: Consider the space $D \cup \{p\}$ where $|D| = 2^\omega$, the elements of D are isolated, and the cocountable subsets of D forms the neighborhood base of p.

Let us recall the following theorem [8, Theorem 4.4.4].

THEOREM: If X is a P-space (i.e., the intersection of countable many open sets is open) with $w(X) \leq 2^\omega$ then X can be embedded into $\beta\omega \setminus \omega$.

According to this theorem the space $D \cup \{p\}$ can be regarded as a subspace of $\beta\omega\backslash\omega$. Consider the subspace $Y = \omega \cup D$ of $\beta\omega$. If U is an uncountable open subset of $\beta\omega$ with $|U \cap Y| > \omega$ then $|U \cap D| > \omega$, so $p \in \overline{U \cap D}$. But ω is dense in $\beta\omega$, so $U \cap D \subset \overline{U \cap \omega}$. Hence, $p \in \overline{U \cap \omega}$, i.e., $U \cap \omega \in p$. (Remember that the elements of $\beta\omega\backslash\omega$ are just the ultrafilters on ω). But the intersection of finitely many elements of p is not empty, so Y is $P_{<\omega}$. ❑

PROPOSITION 2.2: There are first countable, locally compact, scattered $P_{<\omega}$ spaces of size ω_1.

We present two different examples. The first is due to S. Shelah [9].

First proof of Proposition 2.2: The underlying set of our space X will be $\omega_1 \times \omega$. For $x \in \omega_1 \times \omega$ write $x = \langle \pi_0(x), \pi_1(x) \rangle$. For $A \subset \omega_1 \times \omega$ and $i < 2$ put $\pi_i(A) = \pi_i''A$. If $\alpha < \omega_1$ let $Y_\alpha = \{\alpha\} \times \omega$ and $X_\alpha = \alpha \times \omega$.

Let P_0 be the family of pairs $p = \langle A, f \rangle$ which satisfy (i)–(iv) below:

(i) $A \in [\omega_1 \times \omega]^{<\omega}$,

(ii) f is a function, $f: A \times A \to 2$.

To formulate (iii) and (iv) write $U(x) = U^p(x) = \{y \in A : f(y, x) = 1\}$ for $x \in A$.

(iii) $x \in U(x)$, $U(x)\backslash\{x\} \subset X_{\pi_0(x)}$,

(iv) $x \in U(y)$ implies $U(x) \subset U(y)$.

For $p = \langle A, f \rangle$ and $q = \langle B, g \rangle$ from P_0 let

$$\begin{aligned} p \leq_0 q \text{ if and only if } & A \supset B, \\ & f \supset g, \\ & \forall x, y, z_0, \ldots, z_{l-1} \in B \\ & \text{if } U^q(x) \cap U^q(y) = \textstyle\bigcup_{j<l} U^q(z_j) \\ & \text{then } U^p(x) \cap U^p(y) = \textstyle\bigcup_{j<l} U^p(z_j). \end{aligned}$$

For $p \in P_0$ let

$$\mathcal{W}(p) = \{\langle x, s \rangle \in A^p \times [A^p]^{<\omega} : s \subset X_{\pi_0(x)}\}.$$

For $w = \langle x, s \rangle \in \mathcal{W}(p)$ let $b(w) = x$ and $U^p(w) = U^p(x)\backslash \bigcup_{y \in s} U^p(y)$. Let

$$\mathcal{D}(p) = \{\langle w_0, w_1 \rangle \in \mathcal{W}(p) \times \mathcal{W}(p) : \pi_0(b(w_0)) < \pi_0(b(w_1))\}.$$

Now let P be the family of triples $\langle A, f, d \rangle$, where $\langle A, f \rangle \in P_0$, d is a function, $d: \mathcal{D}(\langle A, f \rangle) \to \omega$, such that:

(1) if $d(\langle w_0, w_1 \rangle) = d(\langle w_0', w_1' \rangle)$ then $\pi_0(b(w_1)) \neq \pi_0(b(w_1'))$,

(2) if $d(\langle w_0, w_1 \rangle) = d(\langle w_0', w_1' \rangle)$, $\alpha \in \pi_0(A)$ and $\alpha < \pi_0(b(w_0)) \leq \pi_0(b(w_1'))$ then

$$U^p(w_0) \cap U^p(w_1') \cap Y_\alpha \neq \varnothing.$$

Write $p = \langle A_p, f_p, d_p \rangle$ for $p \in P$. If $p = \langle A, f, d \rangle$ and $q = \langle B, g, e \rangle$ are from P take $p \leq q$ if and only if $\langle A, f \rangle \leq_0 \langle b, g \rangle$ and $d \supset e$. Let $\mathcal{P} = \langle P, \leq \rangle$. ❑

For $\alpha < \omega_1$ let $E_\alpha = \{p \in P : \alpha \in \pi_1(A^p)\}$ and $\mathcal{E} = \{E_\alpha : \alpha \in \omega_1\}$.

LEMMA 2.3: (ZFC) There is an $\mathcal{P}$-generic filter over $\mathcal{E}$.

Proof: This follows from [10], or Velleman's ω-morass (a weak form of Martin's axiom provable in ZFC, see [11]) can be applied for $\mathcal{P}$ and $\mathcal{E}$. We explain the second argument more explicitly. By [11, Theorem 3.4] it is enough to prove that $\mathcal{E}$ is weakly ω-indiscernible for $\mathcal{P}$ (see [11, Definition 1.5]).

For $n < \omega$, $\alpha < \omega_1$ and an order-preserving function $f: n \to \alpha$ define $\sigma_f: P_n \to P_\alpha$ as follows. For $x = \langle \nu, m\rangle \in n \times \omega$ let $\sigma_f(x) = \langle f(\nu), m\rangle$. For $\langle x, s\rangle \in (n \times \omega) \times [n \times \omega]^{<\omega}$; let $\sigma_f(\langle x,s\rangle) = \langle \sigma_f(x), \sigma_f{}''s\rangle$. For $p = \langle A, f, d\rangle \in P$ take $\sigma_f(p) = \langle \sigma_f{}''A, f', d'\rangle \in P$, where $f'(\sigma_f(x), \sigma_f(y)) = f(x, y)$ and $d'(\langle \sigma_f(w), \sigma_f(w')\rangle) = d(\langle w, w'\rangle)$.

We should check conditions [11, Definition 1.5, (1)–(5)]. All but (5) are clear. So let $s < n < \omega$, $f = f(s, n)$ (i.e., $\mathrm{dom}(f) = n$, $f(i) = i$ for $i < s$ and $f(i) = n + i - s$ for $i \geq s$) and $p \in P_n$. We need to find a common extension of p and $q = \sigma_f(p)$.

Let $B = A_p \cup A_q$ and $g: B \times B \to 2$ be the function defined by the stipulation $g^{-1}\{1\} = f_p^{-1}\{1\} \cup f_q^{-1}\{1\}$. Clearly, $r = \langle B, g\rangle$ is a common extension of $p^- = \langle A_p, f_p\rangle$ and $q^- = \langle A_q, f_q\rangle$ in $\mathcal{P}_0$. Now for each $\bar{w}_0 = \langle w_0, w_1\rangle \in \mathcal{D}(p)\backslash\mathcal{D}(q)$, $\bar{w}_1 = \langle w_0', w_1'\rangle \in \mathcal{D}(q)\backslash\mathcal{D}(p)$ and $\alpha \in \pi_0(B)$ with $d_p(\bar{w}_0) = d_q(\bar{w}_1)$ and $\alpha < \pi_0(b(w_0)) \leq \pi_0(b(w_1'))$ pick a different element $z_{\{\bar{w}_0,\bar{w}_1,\alpha\}} \in Y_\alpha\backslash B$. Extend r to $r' = \langle B', g'\rangle \in \mathcal{P}_0$ by adding the points $z_{\{\bar{w}_0,\bar{w}_1,\alpha\}}$ to B in such a way that for any $x \in B$ we have $z_{\{\bar{w}_0,\bar{w}_1,\alpha\}} \in U^{r'}(x)$ if and only if $b(w_0) \in U^r(x)$ or $b(w_1') \in U^q(x)$. Finally, taking $d = d_p \cup d_q$ we choose $d': \mathcal{D}(r') \to \omega$ such that $d' \supset d$, $\mathrm{ran}(d'\backslash d) \cap \mathrm{ran}(d) = \emptyset$ and $d'\backslash d$ is 1–1. Now $r^* = \langle B', g', d'\rangle$ is a common extension of p and $q = \sigma_f(p)$ in P. ❑

Let $\mathcal{G}$ be an $\mathcal{E}$-generic filter over $\mathcal{P}$. The set $A = \bigcup\{A_p : p \in \mathcal{G}\}$ is uncountable because $\pi_0(A) = \omega_1$ by the genericity of $\mathcal{G}$. Take $U(x) = \bigcup\{U^p(x) : x \in A^p\}$ and

$$\mathcal{B}_x = \{U(x)\backslash\textstyle\bigcup_{j<l} U(z_j) : l < \omega, z_0, \ldots, z_{l-1} \in X_{<\pi_0(x)} \cap A\}.$$

Let $X = \langle A, \tau\rangle$, where $\mathcal{B}_x$ is the neighborhood base of x in X.

It is straightforward that X is scattered, 0-dimensional and locally compact.

Finally we show that X is strongly P_2. Let

$$\mathcal{W} = \{\langle x, s\rangle \in A \times [A]^{<\omega} : s \subset X_{\pi_0(x)}\}.$$

For $w = \langle x, s\rangle \in \mathcal{W}$ put $U_w = U(x)\backslash\bigcup_{x' \in s} U(x')$.

Now assume that V_0 and V_1 are uncountable open subsets of X. We can find a sequence $S = \{\langle w_0^\alpha, w_1^\alpha\rangle : \alpha < \omega_1\} \subset [\mathcal{W}]^2$ such that:

(i) $U_{w_i^\alpha} \subset V_i$ for $i < 2$ and $\alpha < \omega_1$,

(ii) $\alpha + 1 = \pi_0(b(w_0^\alpha)) < \pi_0(b(w_1^\alpha))$.

Let $d = \bigcup\{d_p : p \in \mathcal{G}\}$. Then $S \subset \mathrm{dom}(d)$, so there is an uncountable $I \subset \omega_1$ such that d is constant on $\{\langle w_0^\alpha, w_1^\alpha\rangle : \alpha \in I\}$. Then, by the definition of $\mathcal{P}$, we have $U_{w_\alpha^0} \cap U_{w_\beta^1} \cap Y_\alpha \neq \emptyset$ for $\alpha < \beta \in I$. So the intersection V_0 and V_1 is uncountable. ❑

Second proof of Proposition 2.2: The construction will be divided into two parts. First we introduce the notion of α-good spaces and we show that ω_1-good spaces are $P_{<\omega}$. Then in the second part we construct an ω_1-good space in ZFC.

DEFINITION 2.4: Let $X = \langle \nu \times \omega, \tau\rangle$ be a locally compact scattered topological space of height $\nu \leq \omega_1$ such that the α-th level of X is just $X_\alpha = \{\alpha\} \times \omega$. Write $X^{(n)} = \nu \times \{n\}$. We say that X is ν-*good* if and only if:

(a) $X^{(n)}$ is a closed subspace of X and the natural bijection between $X^{(n)}$ and ν is a homeomorphism;
(b) for each limit ordinal $\beta < \nu$ and each $\alpha < \beta$ if $x \in X_\beta$ and U is an open neighborhood of x then the set

$$K_{x,U,\alpha} = \{n \in \omega : U \cap X^{(n)} \cap (\bigcup_{\alpha \le \gamma < \beta} X_\gamma) = \emptyset\}$$

is finite.

LEMMA 2.5: An ω_1-good space X is strongly P_2.

Proof of Lemma 2.5: Write $X_{<\alpha}$ for $\bigcup_{\nu<\alpha} X_\nu$. Let U and V be uncountable open subsets of X. Let

$$I = \{n \in \omega : \{\alpha : \langle \alpha, n\rangle \in U\} \text{ is stationary}\}.$$

By (a) we have $|\bigcup\{X^{(n)} : n \in I\} \setminus U| \le \omega$. Since the ideal NS_{ω_1} is σ-complete, the set $A = \{\alpha < \omega_1 : \exists n \in \omega \setminus I \langle \alpha, n\rangle \in U\}$ is not stationary. Hence, there is $\alpha < \omega_1$ such that $U \cap X_\alpha \subset \{\alpha\} \times I$. But X_α is dense in $X \setminus X_{<\alpha}$, so $U \cap X_\alpha$ is dense in $U \setminus X_{<\alpha}$. It follows that I is infinite.

We show $(U \cap V) \setminus X_{<\alpha} \ne \emptyset$ for each $\alpha < \omega_1$.

Indeed pick $y \in V \cap X_\beta$ where $\alpha < \beta$ is limit. By (b) the set $K_{y,V,\alpha}$ is finite, so we can choose $n \in I \setminus K_{y,V,\alpha}$. Then $V \cap X^{(n)} \cap (X_{<\beta} \setminus X_{<\alpha}) \ne \emptyset$. But $|X^{(n)} \setminus U| \le \omega$, so $V \cap U \cap (X_{<\beta} \setminus X_{<\alpha}) \ne \emptyset$ provided that α is large enough, which completes the proof of the lemma. ❑

So it is enough to construct an ω_1-good space.

LEMMA 2.6: If $\nu < \omega_1$ and $X = \langle \nu \times \omega, \tau\rangle$ is a ν-good space then there is a $\nu + 1$-good space Y such that X is a subspace of Y.

Proof of Lemma 2.6: First recall that X is collectionwise normal because it is countable and regular. During the construction we have to distinguish two cases.

CASE 1: $\nu = \mu + 1$.

Let $\{x_{n,i} : n,i < \omega\}$ be a 1–1 enumeration of $\{\mu\} \times \omega$. The family $\{x_{n,i} : n,i < \omega\}$ is closed and discrete, so applying the collectionwise normality of X we can find a closed discrete family $\{U_{n,i} : n,i < \omega\}$ of clopen subsets of X with $x_{n,i} \in U_{n,i}$. Take $W_{n,i} = \{\langle \nu, n\rangle\} \cup \bigcup\{U_{n,j} : i \le j < \omega\}$. Let the neighborhood base of $\langle \nu, n\rangle$ in Y be $\mathcal{W}_n = \{W_{n,i} : i < \omega\}$.

CASE 2: ν is limit.

Let $\{\gamma_n : n < \omega\}$ be a strictly increasing, cofinal sequence in ν. By induction choose ordinals $\delta_n < \nu$ and distinct points $\{x_{n,i} : n \le i < \omega\} \subset X$ such that:
(i) $\gamma_n < \delta_n$,
(ii) $x_{n,i} \in (\delta_n \setminus \gamma_n) \times \{n\}$.

Put $E_n = (\nu \setminus \delta_n) \times \{n\}$. Then $\mathcal{F} = \{E_n : n \in \omega\} \cup \{x_{n,i} : n \le i \in \omega\}$ is a closed, discrete family of closed sets because for each $\alpha < \nu$ the set $\{F \in \mathcal{F} : F \cap X_{<\alpha} \ne \emptyset\}$ is finite. Applying the collectionwise normality of X we can find a closed, discrete family $\{U_n : n \in \omega\} \cup \{V_{n,i} : n \le i < \omega\}$ of clopen subsets of X such that

$E_n \subset U_n$ and $x_{n,i} \in V_{n,i}$. For each $x = \langle \alpha, m \rangle \in X$ fix a countable neighborhood base $\mathcal{B}_x$ of x containing only clopen subsets of $X_{\leq \alpha}$. The neighborhood base of $\langle \nu, n \rangle$ in Y will be generated by the sets

$$W_{n,j,y,B} = \Big(\bigcup_{j \leq i < \omega} V_{n,i} \Big) \cup (U_n \setminus B)$$

where $n \leq j < \omega$, $y \in U_n$ and $B \in \mathcal{B}_y$. It is easy to see that Y satisfies the requirements. ❑

Now define, by induction on ν, ν-good spaces X_ν for $\nu \leq \omega_1$ which extend each other. For successor ν we can apply Lemma 2.6. For limit ν we can simply take the union. ❑

3. Ψ-LIKE SPACES

All examples of locally compact P_2 spaces in [4, Theorems 3, 4, and 8] are Ψ-like. In this section we will see why the examples of first countable, locally compact $P_{<\omega}$ spaces constructed in the previous section in ZFC are not Ψ-like.

THEOREM 3.1: If MA_{ω_1} (σ-linked) holds then there is no Ψ-like P_3-space.

Theorem 3.1 follows from the following combinatorial result.

THEOREM 3.2: (MA_κ(σ-linked)) If $\{F_\alpha \colon \alpha < \kappa\} \subset [\omega]^\omega$ is an almost disjoint family then ω and κ have partitions (a_0, a_1, a_2) and (I_0, I_1, I_2), respectively, with $|a_i| = \omega$ and $|I_i| = \kappa$ such that

$$\forall i \in 3 \ \ \forall \alpha \in I_i \ \ |F_\alpha \cap a_i| < \omega. \tag{*}$$

REMARK: As the referee pointed out, MA(σ-centered) does not imply this statement because the strong Luzin property (see Definition 3.8 below) of an almost disjoint family is preserved by any σ-centered forcing. As it was recalled in [4], partitioning ω into two pieces is also not enough: a Luzin gap , i.e., an almost disjoint family $\{B_\beta \colon \beta < \omega_1\} \subset [\omega]^\omega$ such that for each $B \in [\omega]^\omega$ either the set $\{\beta < \omega_1 \colon |B_\beta \cap B| < \omega\}$ or the set $\{\beta < \omega_1 \colon |B_\beta \setminus B| < \omega\}$ is at most countable, can be constructed in ZFC.

Proof of Theorem 3.2: For $I \subset \kappa$ write $F_I = \bigcup_{\alpha \in I} F_\alpha$. Define the poset $\mathcal{P} = \langle P, \leq \rangle$ as follows. Its underlying set P consists of 7-tuples $\langle m, A_i, I_i \colon i \in 3 \rangle$ satisfying (i)–(iii) below:

(i) $m \in \omega$, $A_i \in [\omega]^{<\omega}$, $I_i \in [\kappa]^{<\omega}$,

(ii) (A_0, A_1, A_2) is a partition of m,

(iii) $F_{I_i} \cap F_{I_j} \subset m$ for $i \neq j$.

Write $p = \langle m^p, A_i^p, I_i^p \colon i \in 3 \rangle$ and $I^p = \bigcup_{i \in 3} I_i^p$ for $p \in P$. Given $p, q \in P$ we set $p \leq q$ if and only if

(a) $m^q \leq m^p$,

(b) $A_i^q = A_i^p \cap m^q$ for $i \in 3$,

(c) $I_i^q \subset I_i^p$ for $i \in 3$,

(d) $F_{I_i^q} \cap (A_i^p \setminus A_i^q) = \varnothing$ for $i \in 3$.

Obviously $\leq$ is a partial order on P.

For $m \in \omega$ and $i \in 3$ put

$$D_{m,i} = \{p \in P: A_i^p \setminus m \neq \emptyset\}$$

and

$$D_m = \{p \in P: m^p \geq m\}.$$

LEMMA 3.3: $\forall m \in \omega\ \forall i \in 3\ \forall q\ \exists p \in D_{m,i}\ p \leq q \wedge I^p = I^q$.

Proof: Let $q \in P$ with $A_i^q \subset m$. We can assume that $m \geq m^q$. Pick $m^* \in (\omega \setminus F_{I_i^q}) \setminus m$. Choose a function f: $[m^q, m^*] \to 3$ such that $f(m^*) = i$ and $k \notin F^q_{I_{f(k)}}$ for $k \in [m^q, m^*)$. Such a function exists by (iii). Let $A_j = A_j^q \cup f^{-1}\{j\}$ for $j < 3$ and $p = \langle m^* + 1, A_j, F_j^q: j \in 3\rangle$. Obviously $p \in P$ and m^* witnesses $p \in D_{m,i}$. Moreover, $p \leq q$. Indeed, conditions (a)–(c) are trivial and (d) holds because $A_j^p \setminus A_j^q = f^{-1}\{j\}$. ❑

LEMMA 3.4: $\forall m \in \omega\ \forall q\ \exists p \in D_m\ p \leq q \wedge I^p = I^q$.

Proof: Straightforward by Lemma 3.3. ❑

LEMMA 3.5: If $\alpha \notin I^q$ then for each $i \in 3$ there is a $p \leq q$ with $\alpha \in I_i^p$.

Proof: We can assume that $i = 0$. Fix $m \in \omega$ with $F_\alpha \cap F_{I^q} \subset m$. Using Lemma 3.4 pick $r \in D_m$ such that $r \leq q$ and $I^q = I^r$. Put $p = \langle m^r, A_j^r: j \in 3, I_0^r \cup \{\alpha\}, I_1^r, I_2^r\rangle$. Then $p \in P$ because (iii) holds by $F_\alpha \cap F_{I^p \setminus \{\alpha\}} = F_\alpha \cap F_{I^q} \subset m \subseteq m^r = m^p$. Since $p \leq q$ is trivial, the lemma is proved. ❑

For $\alpha \in \kappa$ put

$$E_{\alpha,i} = \{p \in P: I_i^p \cap [\omega\alpha, \omega\alpha + \omega) \neq \emptyset\}$$

and

$$E_\alpha = \{p \in P: \alpha \in I^p\}.$$

LEMMA 3.6: Both $E_{\alpha,i}$ and E_α are dense in $\mathcal{P}$.

Proof: Straightforward by Lemma 3.5. ❑

LEMMA 3.7: $\mathcal{P}$ is σ-linked.

Proof: By $MA_{\omega_1}(\sigma\text{-linked})$ we have $\kappa < 2^\omega$ and so we can choose a countable dense set D in the product space 3^κ. For each $p = \langle m, A, I_i: i \in 3\rangle \in \mathcal{P}$ pick $d_p \in D$ such that $d_p(\alpha) = i$ whenever $i \in 3$ and $\alpha \in I_i$.

Let p_0 and p_1 be elements of P, $p_j = \langle m^j, A_i^j, I_i^j: i \in 3\rangle$. We show that if

(1) $m^0 = m^1$,

(2) $A_i^0 = A_i^1$ for $i \in 3$,

(3) $d_{p_0} = d_{p_1}$,

then p_0 and p_1 are compatible. Clearly this statement yields that $\mathcal{P}$ is σ-linked.

Let $I^j = \bigcup_{i \in 3} I_i^j$ for $j \in 2$ and $I = I^0 \cap I^1$. Fix first $m' \geq m$ with

$$F_{I^0 \setminus I} \cap F_{I^1 \setminus I} \subset m'. \tag{†}$$

If $k \in [m, m')$ and $j \in 2$ then, by (iii), there is at most one $i_j(k) \in 3$ with $k \in F_{I^j_{i_j(k)}}$. Choose $g(k) \in 3 \setminus \{i_0(k), i_1(k)\}$. Then

$$k \notin F_{I^0_{g(k)}} \cup I^1_{g(k)}.$$

Define the partition (A_0', A_1', A_2') of m' by the equations $A_i' = A_i \cup g^{-1}\{i\}$ for $i < 3$.

Put $p = \langle m', A_i', I_i^0 \cup I_i^1 : i \in 3\rangle$.

Check first $p \in P$. Conditions (i) and (ii) are trivial. Since $I_i^0 \cap I = I_i^1 \cap I$ by (3) we have

$$F_{I_i^0 \cup I_i^1} \cap F_{I_j^0} =$$

$$(F_{I_i^0 \cap I} \cap F_{I_j^0}) \cup (F_{I_j^1 \setminus I} \cap F_{I_j^1}) \cup (F_{I_i^0 \setminus I} \cap F_{I_j^0 \setminus I}) \subset m \cup m \cup m' = m'$$

by (iii) for p_0 and p_1 and by (†). Thus (iii) holds for p. Finally we show that p is a common extension of p_0 and p_1. Conditions (a), (b), and (c) obviously hold. But $A_i^p \setminus A_i^{p_0} = A_i' \setminus A_i = g^{-1}\{i\}$ and $g^{-1}\{i\} \cap F_{I_i^0 \cup I_i^0} = \varnothing$ by the choice of g, so (d) is also satisfied. The lemma is proved. ❑

We are now ready to conclude the proof of the theorem. Let

$$\mathcal{D} = \{D_m, D_{m,i} : m \in \omega,\ i \in 3\} \cup \{E_\alpha, E_{\alpha,i} : \alpha \in \kappa,\ i \in 3\}.$$

By Lemmas 3.3–3.6, $\mathcal{D}$ is a family of dense subsets of $\mathcal{P}$. Since $\mathcal{P}$ is σ-linked, MA_κ(σ-linked) implies that there is a $\mathcal{D}$-generic filter $\mathcal{G}$ over $\mathcal{P}$. Put $a_i = \bigcup\{A_i^p : p \in \mathcal{G}\}$ and $I_i = \bigcup\{I_i^p : p \in \mathcal{G}\}$. Then (ii) and (b) imply $a_i \cap a_j = \varnothing$ for $i \neq j$. Since $D_m \cap \mathcal{G} \neq \varnothing$, we have $a_0 \cup a_1 \cup a_2 \supset m$. Moreover, $D_{m,i} \cap \mathcal{G} \neq \varnothing$ implies $a_i \not\subset m$. Thus (a_0, a_1, a_2) is a partition of ω into infinite pieces. Similar arguments show that (I_0, I_1, I_2) is a partition of κ into subsets of size κ.

Finally let $i \in 3$ and $\alpha \in I_i$. Pick $q \in \mathcal{G}$ with $\alpha \in I_i^q$. Then

$$F_\alpha \cap a_i \subset F_{I_i^q} \cap a_i = \bigcup_{p \in \mathcal{G}, p \leq q} (F_{I_i^q} \cap A_i^p) \subset A^q \subset m^q$$

by (d). ❑

By the previous theorem the existence of a Ψ-like P_3 space of size ω_1 is not provable in ZFC. On the other hand, we will see that it is also consistent that 2^ω is arbitrarily large and there is a Ψ-like $P_{<\omega}$-space of size 2^ω.

We start with a definition.

DEFINITION 3.8: Let $\mathcal{A} \subset [\omega]^\omega$ be an almost disjoint family. We say that $\mathcal{A}$ is a *strong Luzin gap* if and only if given finitely many uncountable subsets $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ of $\mathcal{A}$ we have that $\bigcap_{i<n}(\bigcup \mathcal{A}_i)$ is infinite.

It is easy to see that $\mathcal{A}$ is a strong Luzin gap if and only if the corresponding Ψ-like space $X[\mathcal{A}]$ (see Definition 1.2) is $P_{<\omega}$. In [4, Theorem 6] a Luzin gap was obtained by a c.c.c forcing due to Hechler [5].

We show that this almost disjoint family is in fact strongly Luzin. First we recall some notations and definitions from [4]. Let $\kappa > \omega$ be a fixed regular cardinal. Write $\mathcal{D} = [\kappa]^{<\omega} \times \kappa$. Let

$$P = \{p : p \text{ is a function, } \mathrm{dom}(p) \in \mathcal{D},\ \mathrm{ran}(p) \subset 2\}$$

If $p, p' \in P$, $\mathrm{dom}(p) = A \times n$, $\mathrm{dom}(p') = A' \times n'$ put $p \le p'$ if and only if $p \supset p'$ and for each $k \in n \setminus n'$ we have $|\{\alpha: p(\alpha,k) = 1\}| = 1$.

If $\mathcal{G}$ is P-generic then let $A_\alpha = \{k: \exists p \in \mathcal{G}\ p(\alpha, k) = 1\}$. Take $\mathcal{A}[\mathcal{G}] = \{A_\alpha: \alpha < \kappa\}$.

LEMMA 3.9: $\mathcal{A}[\mathcal{G}]$ is a strong Luzin gap in V^P.

Proof: Assume on the contrary, that n and m are natural numbers, $p \Vdash$ "$\{\dot{v}_i : i < n\}$ are injective functions from ω_1 to κ and

$$\left(\bigcap_{i<n} \bigcup_{\alpha<\omega_1} A_{\dot{v}_i(\alpha)}\right) \subset m." \tag{*}$$

For each $\alpha < \omega_1$ pick $p_\alpha \le_P p$ and $v_{\alpha,i} \in \omega_1$ for $i < n$ such that

$$p_\alpha \Vdash \text{“}\dot{v}_i(\alpha) = \hat{v}_{\alpha,i} \text{ for each } i < n.\text{”}$$

Without loss of generality the $v_{\alpha,i}$ are pairwise different. Let $\mathrm{dom}(p_\alpha) = D_\alpha \times k_\alpha$. We can assume that k_α are independent of α, $k_\alpha = k \ge m$, $\{D_\alpha: \alpha < \omega_1\}$ forms a Δ-system with kernel D, $|D_\alpha| = |D_\beta|$, $\{v_{\alpha,i}: i < n\} \subset D_\alpha \setminus D$ and denoting by $\sigma_{\alpha,\beta}$ the unique order preserving bijection between D_α and D_β we have that $\sigma_{\alpha,\beta}$ gives an isomorphism between p_α and p_β and $\sigma_{\alpha,\beta}(v_{\alpha,i}) = v_{\beta,i}$ for $i < n$.

Let $\alpha_0 < \ldots\ \alpha_{n-1} < \omega_1$. Define the condition $q \in P$ as follows:

(1) $\mathrm{dom}(q) = (\bigcup_{i<n} D_{\alpha_i}) \times k + 1$,

(2) $q \supset \bigcup_{i<n} p_{\alpha_i}$,

(3) $q(v_{\alpha_i, i}, k) = 1$ for $i < n$,

(4) $q(\xi, k) = 0$ provided $\xi \in \bigcup_{i,n} D_{\alpha_i} \setminus \{v_{\alpha_0 0}, \ldots, v_{\alpha_{n-1}, n-1}\}$.

Then q is a condition which extends p_{α_i} for $i < n$, but $q \Vdash k \in \bigcap_{i<n} A_{v_{\alpha_i,i}}$ which contradicts (*) because $k \ge m$. This concludes the proof of the lemma. ❑

This lemma yields the following corollary.

COROLLARY 3.10: If ZF is consistent then so is ZFC + $2^\omega = \kappa$ is as large as you wish + there is a Ψ-like $P_{<\omega}$ space of size 2^ω.

4. A POSITIVE PARTITION THEOREM BELOW 2^ω

In this section we present the proof of a theorem of Szentmiklóssy which was announced and applied in [4].

Given $A, B \subset \kappa$ we denote by $[A; B]$ the set $\{\{\alpha, \beta\}: \alpha \in A, \beta \in B, \alpha < \beta\}$.

DEFINITION 4.1: Given cardinals κ, λ, and μ the partition relation $\kappa \to ((\lambda; \lambda))^2_\mu$ holds if and only if $\forall f: [\kappa]^2 \to \mu\ \exists A, B \in [\kappa]^\lambda\ \exists \xi < \mu\ \big(\mathrm{tp}(A) = \mathrm{tp}(B) = \lambda \wedge \sup A \le \sup B \wedge f''[A; B] = \{\xi\}\big)$.

THEOREM 4.2: If ZF is consistent then so is ZFC + $2^\omega = \kappa \ge \omega_3$ + $\omega_3 \to ((\omega_1; \omega_1))^2_{\omega_1}$.

The proof is based on the following lemma. To simplify its formulation we write $P = \mathrm{Fn}(\omega_3, \omega_1; \omega_1)$. For $f, g, h \in P$ we write $f = g \dot{\cup} h$ to means that $f = g \cup h$ and $g \cap h = \varnothing$.

LEMMA 4.3: Assume GCH. If $f: [\omega_3]^2 \to P$ then there is some $C \in [\omega_3]^{\omega_2}$ and there are elements $\{p, p(\xi), q(\eta), r(\xi, \eta): \xi, \eta \in C, \xi < \eta\} \subset P$ with the following properties:

(i) $\forall \xi < \eta \in C\ f(\xi, \eta) = p \dot{\cup} p(\xi) \dot{\cup} q(\eta) \dot{\cup} r(\xi, \eta)$,
(ii) $\forall \xi \in C\ \{f(\xi, \eta): \eta \in C \backslash \xi + 1\}$ forms a Δ-system with kernel $p \dot{\cup} p(\xi)$,
(iii) $\forall \eta \in C\ \{f(\xi, \eta): \xi \in C \cap \eta\}$ forms a Δ-system with kernel $p \dot{\cup} q(\eta)$,
(iv) the $p(\xi)$ have disjoint supports,
(v) the $q(\eta)$ have disjoint supports.

Proof of Lemma 4.3: Fix a large enough regular cardinal τ and let N be a countable elementary submodel of $\mathcal{H}(\tau) = \langle H(\tau), \in, < \rangle$ with $f \in N$, where $H(\tau)$ is the family of the sets whose transitive closure has cardinality $< \tau$.

Let $\mathcal{G} = \{g \in N: g$ is a function from $[\omega_3]^{<\omega}$ to $\omega_1\}$ and $\mathcal{H} = \{h \in \mathrm{N}: h$ maps $[\omega_3]^2$ to $\omega_1\}$.

SUBLEMMA 4.3.1: There is $D \subset \omega_3$ of order type $\omega_2 + 1$ which is end-homogeneous for all $g \in \mathcal{G}$ and homogeneous for all $h \in \mathcal{H}$.

Proof of Sublemma 4.3.1: By $2^{\omega_1} = \omega_2$ we can choose an increasing sequence $\langle N_\alpha: \alpha \leq \omega_2 \rangle$ of elementary submodels of $\mathcal{H}(\tau)$ such that $N \in N_0$, $|N_\alpha| = \omega_2$ and $[N_\alpha]^{\omega_1} \subset N_\alpha$.

Since $\mathcal{G}, \mathcal{H} \in N_0$, they have enumerations $\overline{\mathcal{G}} = \langle g_n: n \in \omega \rangle$ and $\overline{\mathcal{H}} = \langle h_n: n \in \omega \rangle$ in it.

Pick an arbitrary $x \in \omega_3 \backslash \sup(N_{\omega_2} \cap \omega_3)$.

By transfinite recursion on α, define an increasing sequence $\{x_\alpha: \alpha < \omega_2\} \subset \omega_3$ with $x_\alpha \in N_\alpha$ such that

$$\forall n \forall a \in [\{x_\beta: \beta < \alpha\}]^{<\omega}\ g_n(a, x_\alpha) = g_n(a, x). \tag{*}$$

This can be done by $[N_\alpha]^{\omega_1} \subset N_\alpha \prec \mathcal{H}(\tau)$ and $x \in \omega_3 \backslash N_\alpha$.

Color the elements of $\{x_\alpha: \alpha < \omega_2\}$ with the $\mathcal{H}$-colors of the pair $\{x_\alpha, x\}$:

$$F(x_\alpha) = \langle h_n(x_\alpha, x): n < \omega \rangle.$$

The range of F has cardinality $\leq \omega_1^{\omega} = \omega_1$ by CH, so there is an F-homogeneous $D' \subset \{x_\alpha: \alpha < \omega_2\}$ of size ω_2, and it is easily seen that $D = D' \cup \{x\}$ satisfies the requirement of the sublemma. ❑

To simplify our notation we will assume that D is just $\omega_2 + 1$.

SUBLEMMA 4.3.2: $\forall \xi < \omega_2\ \exists \delta(\xi) < \omega_2\ \{f(\xi, \eta): \eta \in \omega_2 + 1 \backslash \delta(\xi)\}$ forms a Δ-system.

Proof of Sublemma 4.3.2: For $\xi < \eta < \eta' \leq \omega_2$ let $d(\xi, \eta, \eta') = \mathrm{dom}(f(\xi, \eta)) \cap \mathrm{dom}(f(\xi, \eta))$. Then $d(\xi, \eta, \eta')$ is one of the ω_1 many countable subsets of $\mathrm{dom}(f(\xi, \eta))$, so, by the end-homogeneity, $d(\xi, \eta, \eta')$ is independent of η' for $\xi < \eta < \eta' \leq \omega_2$. Denote this common value by $h(\xi, \eta)$. Unfortunately we can't apply the end-homogeneity for $h(\xi, \eta)$, because its range may have large cardinality. But we can argue in the following way. We know that $\eta < \eta'$ implies $h(\xi, \eta) \subset h(\xi, \eta)'$, because for any $\eta' < \eta''$ we have $\mathrm{dom}(f(\xi, \eta)) \cap \mathrm{dom}(f(\xi, \eta'')) = h(\xi, \eta) = \mathrm{dom}(f(\xi, \eta)) \cap \mathrm{dom}(f(\xi, \eta'))$, so $h(\xi, \eta) \subset \mathrm{dom}(f(\xi, \eta')) \cap \mathrm{dom}(f(\xi, \eta'')) = h(\xi, \eta')$.

But $h(\xi, \eta)$ is a countable set, so

$$\forall \xi < \omega_2 \;\; \exists \delta(\xi) < \omega_2 \;\; \forall \eta, \eta' \geq \delta(\xi) \;\; h(\xi, \eta) = h(\xi, \eta').$$

But this means that $\{\mathrm{dom}(f(\xi, \eta)) \colon \eta > \delta(\xi)\}$ forms a Δ-system with kernel $h(\xi, \delta(\xi))$. But $f(\xi, \eta') \lceil h(\xi, \delta(\xi))$ is independent of η' by the end-homogeneity, so the functions $f(\xi, \eta')$ are pairwise compatible for $\delta(\xi) \leq \eta' \leq \omega_2$. ❑

Applying in four steps this sublemma, a Δ-system argument, transfinite recursion and a Δ-system argument again, we can find a set $D \in [\omega_2 + 1]^{\omega_2}$; with $\omega_2 \in D$ such that

(1) $\forall \xi \in D \; \{f(\xi, \eta) \colon \eta \in D \setminus \xi + 1\}$ forms a Δ-system with kernel $p^{\Delta}(\xi)$. We write $f(\xi, \eta) = p^{\Delta}(\xi) \dot\cup r + (\xi, \eta)$,

(2) $\{p^{\Delta}(\xi) \colon \xi \in D\}$ forms a Δ-system with kernel p. We write $p^{\Delta}(\xi) = p \dot\cup p(\xi)$. So $f(\xi, \eta) = p \dot\cup p(\xi) \dot\cup r^{+}(\xi, \eta)$.

(3) $\forall \eta \in D \; \forall \xi, \xi', \eta' \in D \cap \eta$ if $\xi' < \eta'$ then

$$\big(\mathrm{dom} p(\xi) \cup \mathrm{dom} r^{+}(\xi, \eta)\big) \cap \mathrm{dom}\, f(\xi', \eta') = \varnothing,$$

(4) $\{f(\xi, \omega_2) \colon \xi \in D\}$ forms a Δ-system.

By the end-homogeneity of D, it follows that $\{f(\xi, \eta) \colon \xi \in D \cap \eta\}$ also forms a Δ-system with some kernel $q^{\Delta}(\xi)$ for all $\eta \in D$. Write $q^{\Delta}(\xi) = p \cup q(\eta)$.

Consider the increasing enumeration $\{\delta_{\nu} \colon \nu < \omega_2\}$ of D and let $C = \{\delta_{\nu} \colon \nu$ is limit$\}$.

Claim 1: If $\xi' < \eta' < \xi < \eta$ are from D, then $\mathrm{dom}\, f(\xi', \eta') \cap \mathrm{dom} f(\xi, \eta) = \mathrm{dom}(p)$.

Indeed, $f(\xi, \eta) = p \dot\cup p(\xi) \dot\cup r^{+}(\xi, \eta)$ and both $\mathrm{dom}(p(\xi))$ and $\mathrm{dom} r^{+}(\xi, \eta)$ are disjoint from $\mathrm{dom}(f(\xi', \eta'))$ by (3).

Claim 2: If $\xi' < \eta' < \eta$ are from C, then $\mathrm{dom} f(\xi', \eta') \cap \mathrm{dom} q(\eta) = \varnothing$.

Indeed, pick $\xi \in D \cap (\eta', \eta)$, observe $q(\eta) \subset f(\xi, \eta)$, and apply Claim 1.

Claim 3: If $\eta' < \eta$ are from C, then $\mathrm{dom}(q(\eta')) \cap \mathrm{dom}(q(\eta)) = \varnothing$.

Let $\xi' \in D \cap \nu'$. Then $q(\eta') \subset f(\xi', \eta')$ and apply Claim 2.

Claim 4: If $\xi < \eta$ are from C, then $\mathrm{dom}(p(\xi)) \cap \mathrm{dom}(q(\eta)) = \varnothing$.

Indeed, pick $\eta^* < \xi^* \in D \cap (\xi, \eta)$. Then $\mathrm{dom}(p(\xi)) \cap \mathrm{dom}(q(\eta)) \subset \mathrm{dom}\, f(\xi, \eta^*) \cap \mathrm{dom}(f(\xi^*, \eta) \setminus p)) = \varnothing$ by Claim 1.

So if you take $r(\xi, \eta) = f(\xi, \eta) \setminus \big(p(\xi) \cup q(\eta) \cup p\big)$ for $\xi < \eta \in C$, then the set C and the conditions $\{p, p(\xi), q(\eta), r(\xi, \eta) \colon \xi, \eta \in C, \xi < \eta\}$ satisfy (i)–(v). The lemma is proved. ❑

Proof of Theorem 4.2: We start with a model of ZFC + GCH and fix an arbitrarily large regular cardinal $\kappa \geq \omega_3$. Let $C_{\omega_1} = \mathrm{Fn}(\omega_1, 2, \omega)$ and consider the poset $Q = \prod^{\kappa}_{\omega_1, \omega} C_{\omega_1} = \langle \mathrm{Fn}(\kappa, C_{\omega_1}; \omega_1), \leq_{\omega} \rangle$ where $f \leq_{\omega} g$ if and only if $\mathrm{dom}(f) \supset \mathrm{dom}(g)$, $f(\alpha) \leq_{C_{\omega_1}} g(\alpha)$ for each $\alpha \in \mathrm{dom}(g)$ and $|\{\alpha \in \mathrm{dom}(g) \colon f(\alpha) \neq g(\alpha)\}| < \omega$ (see [2]). We will show that the model V^{Q} satisfies our requirements.

It is known that $V^Q \models 2^\omega = \kappa$, Q is proper and the cardinals in V and in V^Q are the same (see[2]). Obviously the next lemma concludes the proof.

LEMMA 4.4: $V^Q \models$ "$\forall g: [\omega_3]^2 \to \omega_1\ \exists A,B \subset \omega_3\ \mathrm{tp}(A) = (B) = \omega_2$, $\sup A = \sup B$ and $\exists \alpha \in \omega_1\ \forall S \in [A]^\omega\ \exists B_S \in [B]^{\omega_2}$; $g''[S, B_S] = \{\alpha\}$."

Proof of Lemma 4.4: Assume $\dot{g}$ is a name of a function from $[\omega_3]^2$ to ω_1.

For each $\xi < \eta < \omega_3$ pick a condition $s(\xi, \eta) \in Q$ and an ordinal $\alpha(\xi, \eta) \in \omega_1$ with $s(\xi, \eta) \Vdash$ "$\dot{g}(\xi, \eta) = \alpha(\xi, \eta)$." We can assume that $\mathrm{dom}(s(\xi, \eta)) \subset \omega_3$.

Fix an enumeration of C_{ω_1} in V, $\{s_\nu : \nu < \omega_1\}$, and define a bijection F between Q and $\mathrm{Fn}(\omega_3, \omega_1; \omega_1)$ as follows:

$$F(s) = h \text{ if and only if } \mathrm{dom}(h) = s$$
$$\text{and } s(\nu) = c_{h(\nu)} \text{ for all } \nu \in \mathrm{dom}\,(h).$$

Now consider the function $f: [\omega_3]^2 \to P$ defined by the formula $f(\xi, \eta) = F(s(\xi, \eta)) \times \{\langle \omega_3, \alpha(\xi, \eta)\rangle\}$. Formally, the range of f is $P \times (\omega_3 \times \omega_1) = \mathrm{Fn}(\omega_3 + 1, \omega_1; \omega_1)$, but this poset is isomorphic to P. Applying Lemma 4.3 we can get a $C \in [\omega_3]^{\omega_2}$ and elements $\{p, p(\xi), q(\eta), r(\xi, \eta) : \xi,\eta \in C, \xi < \eta\} \subset Q$ and $\alpha \in \omega_1$ satisfying (i)–(v) above and $\alpha(\xi, \eta) = \alpha$ for each $\xi < \eta \in C$. Let $\dot{A}$ and $\dot{B}$ be Q-names of subsets C such that for all $\xi \in C$ we have $[\![\xi \in \dot{A}]\!] = F^{-1}(p \cup p(\xi))$ and $[\![\eta \in \dot{B}]\!] = F^{-1}(p \cup q(\eta))$. It is clear that $F^{-1}(p) \Vdash |\dot{A}| = |\dot{B}| = \omega_2$.

Let S be a countable subset of A in V^Q. Since Q is proper, there is a countable T in V with $S \subset T$.

Let $B^* = \{\eta \in B : g''[T \cap A, \{\eta\}] = \{\alpha\}\}$ and $\dot{B}^*$ be a name for this set.

It is enough to show that $F^{-1}(p) \Vdash |B^*| = \omega_2$. Assume on the contrary that $r \leq F^{-1}(p)$, $\rho \in C$ and $r \Vdash$ "$B^* \subset B \cap \rho$," i.e., B^* is bounded in B.

Let $E = \mathrm{dom}(p) \cup \mathrm{dom}(r) \cup \bigcup\{\mathrm{dom}(p(\xi)) : \xi \in T\}$. Pick $\sigma \in C \backslash \rho$ such that $(\mathrm{dom}(q(\sigma)) \cup \mathrm{dom}(r(\xi, \sigma))) \cap E = \varnothing$ for each $\xi \in T$. Since $r \Vdash |S| = \omega$, there is $\xi \in T$ such that r and $p \cup p(\xi)$ is compatible.

Let $r^* = r \wedge (p \cup p(\xi) \wedge (q(\sigma) \cup r(\xi,\sigma))$. Then r^* forces a contradiction. ❑

This completes the proof of the theorem. ❑

5. MODELS WITHOUT LARGE FIRST COUNTABLE P_2-SPACES

Hajnal and Juhász [4, Problem 10] asked if it is consistent to assume that $2^\omega \geq \omega_2$ and every first countable (or compact) space with property P_2 has cardinality $\leq \omega_1$. We give an affirmative answer here. We will argue in the following way. First we quote the definition of principle $C(\kappa)$ from [7], then we show that $C(\omega_2)$ implies that every first countable P_2 space has cardinality $\leq \omega_1$, finally we will cite a theorem from [7] saying that $C(\omega_2)$ is consistent with any cardinal arithmetic.

DEFINITION 5.1: (See [7].) Let κ be an infinite cardinal. We say that principle $C(\kappa)$ holds if and only if for each family $\{A(\xi, n), B(\xi, n) : \xi \in \kappa, n \in \omega\} \subset [\omega]^\omega$ either (i) or (ii) below holds:

(i) $\exists C \in [\kappa]^\kappa\ \forall n, m \in \omega\ \forall \xi \neq \zeta \in C\ A(\xi,n) \cap B(\zeta, m) \neq \varnothing$,

(ii) $\exists D,E \in [\kappa]^\kappa\ \exists n, m \in \omega\ \forall \xi \in D\ \forall \zeta \in E\ A(\xi, n) \cap B(\zeta, m) = \varnothing$.

THEOREM 5.2: If $C(\kappa)$ holds then each first countable, separable Hausdorff space X of size κ contains two disjoint open sets U and V of cardinality κ.

Proof: Let S be a countable dense subset of X. For each $x \in X$ fix a neighborhood base $\{U(x, n) : n \in \omega\}$ of x in X. Take $A(x, n) = B(x, n) = U(x,n) \cap S$ and apply $C(\kappa)$. Since X is T_2, there is no $C \in [X]^\kappa$ satisfying 5.1(i). So there are $D, E \in [X]^\kappa$ and $n, m \in \omega$ such that $U(x, n) \cap U(y, m) \cap S = \emptyset$ whenever $x \in D$ and $y \in E$. But S is dense in X, therefore, $U = \bigcup\{U(x, n) : x \in D\}$ and $V = \bigcup\{U(y, m) : y \in E\}$ are disjoint and of cardinality κ. ❑

It was proved in [7] that starting from a model of CH, after adding λ-many Cohen reals by the poset $P = \mathrm{Fn}(\lambda, 2, \omega)$, we have that $C(\omega_2)$ holds in V^P. Since P_2 spaces are separable as it was observed in the proof of [4, Theorem 1], Theorem 5.2 yields the following corollary.

COROLLARY 5.3: If ZF is consistent then so is ZFC + "2^ω is as large as you wish" + "every first countable P_2 space has cardinality $\leq \omega_1$."

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On the Problem of Weak Reflections in Compact Spaces

MARTIN MARIA KOVÁR[a]

Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Brno, Technická 8
616 69 Brno, Czech Republic

ABSTRACT: In this paper we present, among others, an improvement of Hušek's characterization of the spaces with the weak compact reflection. Our main results are as follows: A topological space has a weak reflection in compact spaces iff the Wallman remainder is finite. If a θ-regular or T_1 space has a weak compact reflection, then the space is countably compact. A noncompact θ-regular or T_1 space which is weakly $[\omega_1, \infty)^r$-refinable, has no weak reflection in compact spaces.

1. INTRODUCTION

In this paper we present the solution of the problem of the characterization of those topological spaces, which have a weak reflection in compact spaces. This problem probably was posed initially by Zdeněk Frolík about 25 years ago. However, about 3 or 4 years ago J. Adámek and J. Rosický presented it again. Now the problem is explicitely stated in [1].

Recall that a compactification $\gamma(X)$ of a topological space X is said to be a *weak reflection of* X in the class of compact spaces if for every compact Y and every continuous mapping $f: X \to Y$ there exists a mapping $g: \gamma(X) \to Y$ continuously extending f. The concept of weak reflection is a generalization of the well-known notion of *reflection*. For the weak reflection, we do not require the uniqueness of the extension of the mapping f. Note that every topological space has a reflection in compact Hausdorff spaces. This fact is an immediate consequence of the properties of the Čech-Stone compactification of the completly regular T_1 modification of the space.

In [1], Adámek and Rosický stated a natural question: whether the class of all compact spaces is also weakly reflective. This problem was answered by M. Hušek in the negative in [4]. Hušek described some spaces having a weak reflection in compact spaces and some spaces which have no weak reflection in compact spaces. He also fully characterized all normal T_1 spaces with a weak compact reflection; they are exactly the spaces with the finite Čech-Stone remainder. However, the general equivalent characterization of spaces with the weak com-

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[a]E-mail: kovar@dmat.fee.vutbr.cz.

pact reflection has remained unknown because of the fact that the Wallman remainder of the space need not necessarily contain an infinite discrete subspace.

The following Hušek's results are our starting point and we repeat them because of completeness:

THEOREM H.1: If the Wallman remainder of a topological space X is finite, then the Wallman compactification of X is the weak reflection of X in compact spaces.

THEOREM H.2: If a topological space X contains an infinite family $\{X_n\}_{n\in N}$ of closed noncompact subsets such that $X_n \cap X_m$ is compact for $n \neq m$, then X has no weak reflection in compact spaces.

COROLLARY H.1: If the Wallman remainder of a topological space X contains an infinite discrete subspace, then X has no weak reflection in compact spaces.

COROLLARY H.2: A normal T_1-space has a weak reflection in compact spaces iff its Čech-Stone remainder is finite.

2. DEFINITIONS AND DENOTATIONS

All covering properties (compactness, countable compactness, etc.) are assumed in their general form, that is *without any separation axiom.* A topological space X is *compact* if every open cover of X has a finite subcover. We say that X is *countably compact*, if every countable open cover of X admits of a finite subcover.

For any set S the cardinality of S is denoted by $|S|$. Let Φ be a family of subsets of X, $x \in X$ and let $\mathrm{ord}(x, \Phi) = |\{F \mid F \in \Phi, x \in F\}|$. Recall that a topological space X is said to be *weakly* $[\omega_1, \infty)^r$*-refinable* if for any open cover Ω, of uncountable regular cardinality, there exists an open refinement which can be expressed as $\bigcup_{\alpha\in A} \Phi_\alpha$ where $|A| < |\Omega|$ and if $x \in X$ there is some $\alpha \in A$ such that $0 < \mathrm{ord}(x, \Phi_\alpha) < |\Omega|$. Remark that the class of weakly $[\omega_1, \infty)^r$-refinable spaces contains the classes of paracompact spaces, metacompact spaces and also a number of their generalizations (para-Lindelöf, σ-para-Lindelöf, screenable, σ-metacompact, meta-Lindelöf, submeta-Lindelöf, submetacompact, weakly θ-refinable, weakly $\delta\theta$-refinable spaces; for more detail, see [2]).

Let X be a topological space. A filter base Φ in X has a θ*-cluster point* $x \in X$ if every closed neighborhood H of x and every $F \in \Phi$ have a nonempty intersection. A topological space X is said to be θ*-regular* [5] if every filter base in X with a θ-cluster point has a cluster point. It is shown in [6] that the class of θ-regular spaces contains all regular, rimcompact, and all paracompact spaces as well.

Finally, recall that the Wallman compactification of X is defined as the set $\omega(X) = X \cup \{y \mid y$ is a nonconvergent ultra-closed filter in $X\}$, where 'ultra-closed' means maximal among all filters, having a base consisting of closed sets. The sets $S(U) = U \cup \{y \mid y \in \omega(X)\backslash X, U \in y\}$, where U is open in X, constitute an open base of $\omega(X)$ (see [3]). Recall that every point of the remainder $\omega(X)\backslash X$ is closed in $\omega(X)$; hence $\omega(X)\backslash X$ is a T_1-space.

3. MAIN RESULTS

We present two main theorems. The first one is the desired equivalent characterization of spaces with the weak compact reflection. In fact, it slightly improves Hušek's results, Theorem H.1 and Corollary H.2. The main idea of that improvement is based on a simple, but important observation: Every infinite T_1 space contains an infinite discrete subspace, or an infinite subspace with the cofinite topology (that is, the topology, where a nonempty set is open iff it has the finite complement). For more detail, the reader is referred to a forthcoming paper [7].

THEOREM A: A topological space X has a weak reflection in compact spaces iff the Wallman remainder of X is finite.

Sketch of Proof: According to Theorem H.1, it remains to prove that if the Wallman remainder $\omega(X)\backslash X$ of X is infinite, the space X has no weak compact reflection. But $\omega(X)\backslash X$ is always a T_1 space; if it is infinite, it should contain an infinite discrete subspace or an infinite subspace with the cofinite topology. The case of an infinite discrete subspace was solved by M. Hušek; however, the case of an infinite subspace with the cofinite topology also allows us to construct an infinite sequence $X_1, X_2, \ldots$, of closed noncompact and pairwise disjoint subsets of X. Now, Theorem H.2 completes the proof. ❑

In the second theorem and its corollary we characterize a relatively large class of spaces which have no weak reflection in compact spaces. For the proof we refer the reader to [7].

THEOREM B: Let X be a topological space having a weak reflection in compact spaces. If X is θ-regular or T_1, then it is countably compact.

We leave to the reader to find an (really trivial) example of a topological T_0 space, which has the finite Wallman remainder, but is not countably compact. Of course, such space is neither T_1 nor θ-regular. Hence the assumptions of axiom T_1 or θ-regularity are substantial.

J.M. Worrell and H.H. Wicke proved that a countably compact weakly $[\omega_1, \infty)^r$-refinable topological space is compact and the proof (see, for instance, [2]) needs no separation axioms. Hence, we have the following corollary:

COROLLARY: Let X be a noncompact topological space which is θ-regular or T_1. If X is weakly $[\omega_1, \infty)^r$-refinable, then it has no weak reflection in compact spaces.

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Free Topological Groups over (Semi) Group Actions

MICHAEL MEGRELISHVILI (LEVY) [a,b]

Department of Mathematics and Computer Science
Bar-Ilan University
52 900 Ramat-Gan, Israel

ABSTRACT: We study equivariant embeddability into G-groups. A new regionally proximal type relation introduced in the paper gives a necessary condition providing some counterexamples. We establish also some sufficient conditions (for locally compact acting semigroups G, for instance) improving results of M.Eisenberg and J. de Vries.

INTRODUCTION

Our aim is to investigate the following question: Let a topological (semi)group G act continuously on a space X. When can X be equivariantly embedded (or at least, G-mapped nontrivially) into a topological group P in such a way that G continuously acts on P by endomorphisms (hence, by automorphisms if G is a group)?

For the particular case when P is a *linear* G-space, see de Vries [11] and the references there.

The question leads us to the definition of the free topological G-group over a (semi)group action (see Definition 1). Our main result is Theorem 6 which enables us to find compact coset G-spaces G/H such that the free topological G-group over G/H is cyclic and discrete (as trivial as possible). Roughly speaking, this means that every continuous G-map of G/H into a G-group P is "collapsed" into a point. This happens, for example, when $G/H = \mathbb{S}^n$ is the n-dimensional sphere where G is the group of all autohomeomorphisms of $\mathbb{S}^n$ endowed with the compact open topology. The main tool will be a new "regionally proximal type" relation (Definition 2) which generalizes the classical notion from topological dynamics.

Eisenberg [3] has shown that if a locally compact group G acts continuously on a Tychonoff space X then the induced "lifted" action $G \times A(X) \to A(X)$ (which is separately continuous for arbitrary G) on the free Abelian topological group $A(X)$ is jointly continuous. A similar result, if $A(X)$ is replaced by the free locally convex space $L(X)$ is also true. This was remarked without proof in [3]. De Vries [11] proved it by categorical methods. We establish that analogous results remain

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[a]Partially supported by the Israel Ministry of Sciences, Grant No. 3505.

[b]E-mail: megereli@bimacs.cs.biu.ac.il.

true for the free topological group $F(X)$ for an arbitrary locally compact topological *semigroup G*.

CONVENTIONS

Recall that a *G-space* or, alternatively, a *topological transformation (semi)group* (abbreviated: *tts, ttg*) is a system $\langle G, X, \alpha\rangle$ in which G is a topologized (semi)group, X is a topological space, and $\alpha\colon G\times X\to X$, $\alpha(g,x)=gx$ is a continuous action. As usual, this means that $(gh)x = g(hx)$ for all $g,h\in G$ and every $x\in X$. If G has an identity e (i.e., if G is a monoid) then we require $ex=x$ for every $x\in X$. A *g-transition* is the mapping $\alpha^g\colon X\to X$, $\alpha^g(x)=gx$ and an *x-orbit mapping* is the mapping $\alpha_x\colon G\to X$, $\alpha_x(g)=gx$. A G-space X will be called a *G-group* or *G-endomorphic* if X is a topological group and each α^g is a group endomorphism. If G is a group, under these circumstances we shall call the G-space X *G-automorphic*. In the case of a linear space X and linear endomorphisms α^g, we obtain the known definition of a *linear G-space* [11].

The filter of all neighborhoods at a point x in a space X is denoted by $N_x(X)$. If μ is a compatible uniformity on a topological space X, then for every $\varepsilon\in\mu$ and $A\subset X$ denote by $\varepsilon(A)$ the set $\{y\in X \mid (x,y)\in\varepsilon,\ x\in A\}$. Subsets A,B will be called *ε-near* if $\varepsilon(A)\cap\varepsilon(B)\neq\varnothing$.

We denote the greatest compatible uniformity by $\mu_{\max}$.

Due to [10], the *left*, *right*, and *upper* uniformities on a topological group will be denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L}\vee\mathcal{R}$, respectively.

We say that an action $\alpha\colon G\times X\to X$ is *locally uniformly equicontinuous* if for every $g\in G$ there exists $V\in N_g(G)$ such that $\{\alpha^g\}_{g\in V}$ is uniformly equicontinuous.

MAIN RESULTS

As usual, for a topological space X denote by $F(X)$, $A(X)$, $L(X)$ the free topological group, the free Abelian topological group, and the free locally convex space, respectively.

DEFINITION 1: Let $\langle G, X, \alpha\rangle$ be a *tts*. We will say that an endomorphic triple $\langle G, F_\alpha(X), \alpha\rangle$ with a continuous G-mapping $i_\alpha\colon X\to F_\alpha(X)$ is the *free topological G-group* over X, if for every continuous G-mapping $\varphi\colon X\to P$ to an endomorphic G-space P there exists an unique continuous G-homomorphism $\tilde\varphi\colon F_\alpha(X)\to P$ such that $\tilde\varphi\circ i_\alpha = \varphi$. If μ is an uniformity on X, then considering uniform G-mappings and the upper uniformities on topological groups, we obtain the definition of the *uniform free topological G-group* over (X,μ). The corresponding universal morphism is denoted by $i_\alpha\colon (X,\mu)\to F_\alpha(X,\mu)$.

The *(uniform) free locally convex G-space* $L_\alpha(X)$ ($L_\alpha(X,\mu)$, respectively) can be defined analogously.

An obvious *equivariant* generalization of the standard product procedure shows that the just defined free G-objects always exist. However, it turns out that the *embedding problem* for i_α is much more complicated. We start with the well-known definition from topological dynamics.

Let $\langle G, X, \alpha\rangle$ be a *tts*, μ a uniformity on X, and $S \subset G$. A pair $(a,b) \in X \times X$ is called *regionally S-proximal* [1] written $(a,b) \in Q_S$, if for every $\varepsilon \in \mu$ and arbitrary neighborhoods $O_1 \in N_a(X)$, $O_2 \in N_b(X)$, there exists $g \in S$ such that gO_1 and gO_2 are ε-near. Otherwise, (a,b) is said to be *regionally S-distal*. The space X is called *regionally S-distal* if $Q_S = \Delta_X := \{(x,x) \mid x \in X\}$. The following definition appears to be new.

DEFINITION 2: (i) We say that a pair $(a,b) \in X \times X$ is *regionally S-pseudoproximal* and write $(a,b) \in Q_S{}^P$ (or: $(a,b) \in Q_S{}^P(X,\mu)$), if there exists a finite $\{a = x_0, x_1, \ldots, x_n = b\}$ with the following property:

($*^S$) for every $\varepsilon \in \mu$ and arbitrary neighborhoods $O_i \in N_{x_i}(X)$, $i \in \{0, 1, \ldots, n\}$ there exists $g \in S$ such that gO_i and gO_{i+1} are ε-near, for every $i \in \{0, 1, \ldots, n-1\}$.

(ii) Let G be a monoid. A pair (a,b) will be called *regionally *-pseudoproximal* if $(a,b) \in Q_V{}^P$ for every $V \in N_e(G)$. This defines a relation $Q_*{}^P = \cap\{Q_V{}^P \mid V \in N_e(G)\}$. If $Q_*{}^P = X \times X$ or $Q_*{}^P = \Delta_X$, then we say that X is *regionally *-pseudoproximal*, or *regionally *-pseudodistal*, respectively.

Obviously, $Q_S{}^P$ and $Q_*{}^P$ are reflexive symmetric relations on X satisfying $Q_S \subset Q_S{}^P$, $Q_*{}^P \subset Q_S{}^P$. In general, $Q_S \neq Q_S{}^P$, and $Q_*{}^P \neq Q_S{}^P$.

EXAMPLE 3: Let $G_n = \{h \in H(I) \mid h(x_i) = x_i, x_i = \frac{i}{n}, i \in \{0, 1, \ldots, n\}\}$ be the topological subgroup of $H(I)$. Consider the *ttg* $\langle G_n, I, \alpha\rangle$ and the canonical uniformity on I. Then, for every natural $n \geq 3$, the elements 0 and 1 are regionally G_n-distal. On the other hand, every pair $(a,b) \in I \times I$ is regionally *-pseudoproximal. In particular, Q_{G_n} is a proper subset of $Q_{G_n}{}^P$ for each $n \geq 3$.

EXAMPLE 4: Define the homeomomorphism $h: I \to I$ by the rule

$$h(x) = \begin{cases} 3x^2, & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} + \frac{1}{3}\sqrt{3x-1}, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x^2 - 4x + 2, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Consider the cyclic group $G = \{h^n\}_{n \in \mathbb{Z}}$ and the natural action $G \times I \to I$. Since $0, \frac{1}{3}, \frac{2}{3}, 1$ are fixed then eventually $(0,1) \notin Q_G$. On the other hand, elementary computations show that $Q_G{}^P = I \times I$. Note also that $Q_*{}^P = \Delta_X$ if G is discrete.

LEMMA 5: If $f: (X_1, \mu_1) \to (X_2, \mu_2)$ is a uniform G-mapping, then $(f \times f)(Q_S{}^P(X_1, \mu_1)) \subset Q_S{}^P(X_2, \mu_2)$ and $(f \times f)(Q_*{}^P(X_1, \mu_1)) \subset Q_*{}^P(X_2, \mu_2)$. In particular, if (X, μ) is regionally *-pseudodistal, then every uniform G-subspace $(Y, \mu|_Y)$ is regionally *-pseudodistal.

THEOREM 6: Let G be a topologized monoid. Then every G-group $\langle G, (X, \mu), \alpha\rangle$ is regionally *-pseudodistal for each $\mu \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$.

Proof: First we consider the case $\mu = \mathcal{R}$. Assuming the contrary, take a pair $(a,b) \in Q_*{}^P$ of distinct elements. Since X is a Hausdorff topological group and α

is continuous, then we can choose neighborhoods $V_0 \in N_e(X)$, $U \in N_e(G)$ such that

$$V_0 \cap g(V_0 ab^{-1}) = \varnothing \quad \forall g \in U. \tag{1}$$

Since $Q_*{}^P \subset Q_U{}^P$, then $(a,b) \in Q_U{}^P$. Consider a finite set $\{x_0, x_1, \ldots, x_n\}$ satisfying Definition 2. Choose symmetric neighborhoods $V_1, V_2 \in N_e(X)$ with the properties:

$$\forall i \in \{0, 1, \ldots, n\} \quad x_0 x_i^{-1} V_2^2 \subset V_1 x_0 x_i^{-1} \tag{2}$$

$$V_1^{n+1} \subset V_0. \tag{3}$$

Due to Definition 2(i), for $\varepsilon := \{(x,y) \in X \times X \mid xy^{-1} \in V_2\}$ we pick an element $g \in U$ such that $g(V_2 x_i)$ and $g(V_2 x_{i+1})$ are V_2-near with respect to the right uniformity $\mathcal{R}$ on X. More precisely, there exist finite sequences $\{p_0, p_1, \ldots, p_{n-1}\}$, $\{q_1, q_2, \ldots, q_n\}$ in V_2 such that $g(p_i x_i)\,(g(q_{i+1} x_{i+1}))^{-1} \in V_2$ for every $i \in \{0, 1, \ldots, n-1\}$.

Since α^g is an endomorphism, then

$$\forall i \in \{0, 1, \ldots, n-1\} \quad g(p_i x_i x_{i+1}{}^{-1} q_{i+1}{}^{-1}) \in V_2. \tag{4}$$

Consider the element $z = g(p_0 x_0 x_1{}^{-1} q_1{}^{-1}) g(p_1 x_1 x_2{}^{-1} q_2{}^{-1}) \cdots g(p_{n-1} x_{n-1} x_n{}^{-1} q_n{}^{-1})$. Since $V_2 \subset V_1$ by (2), then (4) and (3) imply $z \in V_2{}^n \subset V_1{}^n \subset V_0$. Eventually, $z = g(p_0 x_0 x_1{}^{-1}(q_1{}^{-1} p_1) x_1 x_2{}^{-1}(q_2{}^{-1} p_2) \cdots (q_{n-1}{}^{-1} p_{n-1}) x_{n-1} x_n{}^{-1} q_n{}^{-1})$. Clearly, $q_i^{-1} p_i \in V_2{}^{-1} V_2 = V_2{}^2$ for each $i \in \{1, \ldots, n-1\}$. Using (2) and the trivial cancellations of the form $x_0 x_i{}^{-1} x_i x_{i+1}{}^{-1} = x_0 x_{i+1}{}^{-1}$, $(1 \le i \le n-1)$, after $n-1$ steps we get $z \in g(p_0 V_1{}^{n-1} x_0 x_n{}^{-1} q_n{}^{-1}) \subset g(V_1{}^n x_0 x_n{}^{-1} q_n{}^{-1})$. Using (2) (for $i = n$), we obtain $z \in g(V_1{}^{n+1} x_0 x_n{}^{-1}) = g(V_1{}^{n+1} ab^{-1}) \subset g(V_0 ab^{-1})$. Thus, $z \in V_0 \cap g(V_0 ab^{-1})$, which contradicts (1). This proves the case $\mu = \mathcal{R}$.

For $\mu = \mathcal{L}$, use the G-unimorphism $(X, \mathcal{L}) \to (X, \mathcal{R})$, $x \to x^{-1}$ and if $\mu = \mathcal{L} \vee \mathcal{R}$, use Lemma 5 for the uniform G-mapping $f = 1_X\colon (X, \mathcal{L} \vee \mathcal{R}) \to (X, \mathcal{R})$. ❑

THEOREM 7: Let G be a topologized monoid and let (X, μ) be a *-pseudoproximal G-space. Then every uniform G-mapping $(X, \mu) \to (Y, \xi)$ into a G-group Y is constant for each $\xi \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$. In particular, the free uniform G-group $F_\alpha(X, \mu)$ is cyclic and discrete.

Proof: Combine Lemma 5 and Theorem 6. ❑

EXAMPLE 8: Let $X = I^n$ be the n-dimensional cube, or let $X = \mathbb{S}^n$ be the n-dimensional sphere (in both cases $n \in \mathbb{N}$). Denote by $H(X)$ the group of all autohomeomorphisms of X endowed with the compact open topology. Then $\langle H(X), X, \alpha \rangle$ is a regionally *-pseudoproximal *ttg* with respect to the unique uniformity on X. Then, by Theorem 7, the free topological G-group $F_\alpha(X)$ is cyclic and discrete. It is remarkable that, by Effros's Theorem [2], $\mathbb{S}^n$ is a coset space of $H(\mathbb{S}^n)$. If $X = \mathrm{I}$ then the example answers the question posed by the author in [4, Problem 1.14].

QUESTION 9: Under which conditions is the G-space G/H automorphizable (= G-subspace of an automorphic G-space)?

This is so, for example, if H is a *neutral* subgroup. Indeed, in such cases, Theorem 5.8 and Proposition 7.7 from [10] imply that the action α_l of G on G/H is uniformly equicontinuous with respect to the quotient uniformity $\mathcal{L}/H$. Therefore, by [7, Theorem 1.2] (or by our Proposition 12) G/H is even G-linearizable.

QUESTION 10: Under which conditions does the free uniform G-group $F_\alpha(X,\mu)$ coincide with the free uniform group $F(X,\mu)$ over X?

LEMMA 11: Let an action $\alpha\colon G\times X\to X$ be locally uniformly equicontinuous with respect to a uniformity μ on X and let orbit mapping $\alpha_y\colon G\to X$ be continuous for each $y\in Y$, where Y is dense in X. Then α is continuous.

For a stronger version for groups, see [5, Lemma 2.1].

PROPOSITION 12: Let $\alpha\colon G\times X\to X$ be a continuous and locally uniformly equicontinuous action on a uniform space (X,μ). Then $F_\alpha(X,\mu) = F(X,\mu)$ and $L_\alpha(X,\mu) = L(X,\mu)$.

Proof: Let $\tilde{\alpha}\colon G\times F(X,\mu)\to F(X,\mu)$ be the lifted action. Clearly, each g-transition $\tilde{\alpha}^g$ is continuous. Since X algebraically generates $F(X,\mu)$, then the continuity of orbit mappings $\alpha_x\colon G\to X$ and of group operations in $F(X,\mu)$ imply that for each $w\in F(X,\mu)$ the orbit mapping $\tilde{\alpha}_w\colon G\to F(X,\mu)$ is continuous. From the constructive description of a neighborhood system of the identity in $F(X,\mu)$ [8], it follows that V acts $\mathcal{L}\vee\mathcal{R}$-uniformly equicontinuously on $F(X,\mu)$, provided that V acts uniformly equicontinuous on (X,μ). By Lemma 11, $\tilde{\alpha}$ is continuous. Obviously, this implies that $F_\alpha(X,\mu) = F(X,\mu)$. Using [9] we can make essentially the same proof work for $L(X,\mu)$. ❑

Pestov [6] proved the continuity of the associated action $\alpha\colon G\times F_v{}^b(X)\to F_v{}^b(X)$ for the uniformly equicontinuous group action $\alpha\colon G\times X\to X$, where $F_v{}^b(X)$ denotes the free uniform *balanced* (i.e., $\mathcal{L} = \mathcal{R}$) group in a variety v. For an analogous "lifting" Theorem for a modification of the free locally convex spaces, see [7, Theorem 1.2].

LEMMA 13: Every continuous action α of a locally compact topological semigroup G on a Tychonoff space X is locally $\mu_{\max}$-uniformly equicontinuous.

Proof: Let a system $S=\{d_k\}_{k\in K}$ of pseudometrics generate $\mu_{\max}$ and let $\mathbb{B}$ be the system of all compact subsets in G. Consider the family $S^{\mathbb{B}}=\{d_k{}^C \mid k\in K, C\in\mathbb{B}\}$ where $d_k{}^C(x,y)=\sup\{d_k(gx,gy)\mid g\in C\}$. The compactness of C and the continuity of α imply that the system $\{\alpha^g\mid g\in C\}$ is d_k-equicontinuous for every $k\in K$. Then it is easy to see that uniformity θ generated by the system $S^{\mathbb{B}}\cup S$ is compatible with the original topology. If $A,B\in\mathbb{B}$ then $A\cdot B\in\mathbb{B}$. Eventually, the given action is locally θ-uniformly equicontinuous. Finally, observe that the maximality of $\mu_{\max}$ and the inclusion $\mu_{\max}\subset\theta$ imply $\mu_{\max}=\theta$. ❑

THEOREM 14: For every continuous action α of a locally compact topological semigroup G on a Tychonoff space X one has $F_\alpha(X)=F(X)$, $A_\alpha(X)=A(X)$, $L_\alpha(X) = L(X)$.

Proof: It is well known that $F(X,\mu_{\max}) = F(X)$, $A(X,\mu_{\max}) = A(X)$ and $L(X,\mu_{\max})=L(X)$. So we can apply Proposition 12 and Lemma 13. ❑

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Dendrons, Dendritic Spaces, and Uniquely Arcwise Connected Spaces

J. NIKIEL[a]

Department of Mathematics
American University of Beirut
Beirut, Lebanon

ABSTRACT: The paper summarizes recent developments concerning natural orderings and order topologies on uniquely arcwise connected spaces among which dendrons and **R**-trees are given special attention.

This paper is intended as a brief introduction to the theory of connected Hausdorff spaces which admit an acyclic partial ordering related to their topology. Some underlying ideas were developed as early as 1946 by G.S. Young. Further developments are due to A.E. Brouwer, J.J. Charatonik, C. Eberhart, J.K. Harris, J. Lawson, J.C. Mayer, J. van Mill, G.G. Miller, T.B. Muenzenberger, J. Nikiel, L.G. Oversteegen, B.J. Pearson, V.V. Proizvolov, R.E. Smithson, L.E. Ward, Jr., E. Wattel, and many others. A more general theory of pseudo-trees and a quite comprehensive list of references can be found in [10]. For (possibly) different approaches the reader may consult other papers from the list accompanying the present note.

Spaces of interest include dendrons (sometimes called trees in other publications), dendritic spaces (called also tree-like spaces by some authors) and uniquely arc-connected spaces, in particular, arboroids, hyperspaces of subcontinua of hereditarily indecomposable continua, and **R**-trees. Everywhere below *continuum* is an abbreviation for "connected and compact Hausdorff space."

Let $(X, \leq)$ be a partially ordered set. For $x \in X$, we let $L(x) = \{y \in X : y \leq x\}$, $l(x) = \{y : y < x\} = L(x) - \{x\}$, $M(x) = \{y : x \leq y\}$ and $m(x) = \{y : x < y\}$. We shall say that $(X, \leq)$ is a *pseudo-tree* if $L(x)$ is linearly ordered by $\leq$ for each $x \in X$. If, moreover, $L(x)$ is well-ordered for each x, then $(X, \leq)$ is a *tree*.

A topological space X is said to be *orderable* if it admits a linear ordering $\leq$ such that the collection of all intervals of the form either $l(x)$ or $m(x)$ constitutes a subbasis for open sets of X. Ordered continua are called *arcs*. The unique separable arc is homeomorphic to $[0,1]$, it will be denoted by I. We shall denote by J the *long ray*, i.e., a half-open arc originating from the linearly ordered set of all countable ordinals with copies of $]0,1[$ inserted between α and $\alpha+1$ for each $\alpha < \omega_1$.

A space which admits a basis of open sets with finite boundaries will be said to be *rim-finite*.

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[a]E-mail: nikiel@layla.aub.ac.lb.

A space X is *arc-connected* if every two points of X belong to an arc contained in X. And it is *I-connected* if every two points belong to a copy of $I = [0,1]$ in X. Obviously, each metrizable arc-connected space is I-connected.

We shall say that an arc-connected space X is *uniquely arc-connected* if, for every $x \neq y \in X$, there exists exactly one arc K in X which has the end points x and y. That unique arc K will be denoted by $[x,y]$ (also, we let $[x,x] = \{x\}$).

A uniquely arc-connected Hausdorff space X is *nested* if the union of each linearly ordered by inclusion collection of subarcs of X is contained in an arc.

The following are examples of uniquely arc-connected Hausdorff spaces:

(1) the hyperspace $C(Y)$ of all subcontinua of a hereditarily indecomposable continuum Y (with its Vietoris topology);
(2) *arboroids*, i.e., arc-connected hereditarily unicoherent continua (recall that a continuum Z is *hereditarily unicoherent* if the intersection of every two subcontinua of Z is connected); metrizable arboroids are called *dendroids*;
(3) dendrons (see below, dendrons are exactly locally connected arboroids);
(4) **R**-*trees*, i.e., metrizable uniquely arc-connected spaces which admit a metric such that each arc is isometric to an interval of real numbers.

It is well known that spaces in (1)–(3) are nested, while **R**-trees usually are not nested. The "Warsaw circle" is an example of a metrizable uniquely arc-connected continuum which is not nested.

Let X be a uniquely arc-connected Hausdorff space and $p \in X$. We define the *weak cut-point* (partial) *ordering* $\leq_p$ with the *base point* p on X by $x \leq_p y$ if $x \in [p,y]$ (equivalently, if $[p,x] \subset [p,y]$). Quite obviously, $(X, \leq_p)$ is a pseudo-tree.

A *dendritic space* is a connected space every two points of which can be separated by the omission of some third point. It easily follows that each dendritic space is Hausdorff. Compact dendritic spaces are called *dendrons*, and metrizable dendrons are called *dendrites*. Clearly, each arc is a dendron. It is well-known (see, e.g., [7]) that each dendron is uniquely arc-connected, nested, hereditarily unicoherent, locally arc-connected, and rim-finite (whence hereditarily locally connected). In 1974, Brouwer, Cornette, and Pearson gave independent proofs that each dendron is a continuous image of some arc (more general facts are now available). Therefore, dendrons are monotonically normal [3]. About 1984, T. Kimura and Nikiel independently proved that dendrons are regular supercompact.

Let X be a dendritic space and $p \in X$. For $x, y \in X$ we let $[x,y] = \{x,y\} \cup \{z \in X :$ z separates X between x and $y\}$. In particular, $[x,x] = \{x\}$. Let us define the *cut-point* (partial) *ordering* $\leq_p$ with the *base point* p on X by letting $x \leq_p y$ if $x \in [p,y]$. It follows that $(X, \leq_p)$ is a pseudo-tree.

The space $\{(0,0)\} \cup \{(x, \sin\frac{\pi}{x}) : 0 < x \leq 1\}$ with its topology induced from the Euclidean plane is dendritic but not arcwise connected.

In general, if K is an arc in a dendritic space X, then $K = [a,b]$ for some $a, b \in X$. And if a dendritic space is either rim-compact or locally connected then it is arc-connected (see [11]). The following was proved by Proizvolov in 1969:

THEOREM 1: For a dendritic space X the following conditions are equivalent:
(i) X is rim-compact,

(ii) X is rim-finite,

(iii) X admits a dendritic compactification, i.e., it is homeomorphic to a dense subset of a dendron.

Roughly speaking, the dendritic compactification mentioned in (iii) is formed from X by addition of the "missing end points."

The following remarkable fact is due to van Mill and Wattel (see, e.g., [6]).

THEOREM 2: A Hausdorff space X can be embedded into a dendron if and only if it has a subbasis $\mathbf{S}$ such that for every $A, B \in \mathbf{S}$ at least one of the following conditions holds: $A \cup B = X$, $A \cap B = \varnothing$, $A \subset B$ or $B \subset A$.

For a given pseudo-tree $(X, \leq)$ there are two immediate ways of introducing an order compatible topology on X [10]. Both are similar to the construction of the order topology on a linearly ordered set. One of those topologies is the *interval topology* $\mathbf{T}_I$ which is obtained when the family $\{M(x) : x \in X\} \cup \{L(x) : x \in X\}$ is taken as a subbasis for closed sets. The other one is the *weak order topology* $\mathbf{T}_{\leq}$ which has $\{m(x) : x \in X\} \cup \{X - M(x) : x \in X\}$ as a subbasis for open sets. It has much better order compatibility properties than the interval topology. Of course, pseudo-trees are much more complicated than linearly ordered sets. Therefore, also $\mathbf{T}_{\leq}$ does not have all the properties of a well-behaved order topology. For example, $(X, \mathbf{T}_{\leq})$ need not be Hausdorff, maximal linearly ordered subsets of $(X, \leq)$ need not be closed in $(X, \mathbf{T}_{\leq})$ and $\leq$ need not be a *continuous ordering* on $(X, \mathbf{T}_{\leq})$ (i.e., it need not be a closed subset of $X \times X$). All pathologies are removed when the *order topology* $\mathbf{T}'_{\leq}$ on X is introduced by adding to $\mathbf{T}_{\leq}$ all sets of the form $M(x)$, where x is such that, for some $y \neq x$, either $l(x) = l(y)$ or $l(x) = L(y)$ (see [10] for details; $\mathbf{T}_{\leq}$ and $\mathbf{T}'_{\leq}$ coincide if and only if $(X, \mathbf{T}_{\leq})$ is a Hausdorff space). As remarked in [10, 6.12 (iii) and 6.17 (iii)], $\mathbf{T}'_{\leq_p}$ coincides with the Lawson topology for most pseudo-trees $(X, \leq)$. The class of those pseudo-trees includes the ones for which $\mathbf{T}'_{\leq_p} = \mathbf{T}_{\leq_p}$.

Now, let $(X, \mathbf{T})$ be either a dendritic space or a uniquely arc-connected Hausdorff space, and let $p \in X$. Recall that $(X, \leq_p)$ is a pseudo-tree (see [7], [8], [11] or [10] for its order characterization), and so one can consider its order topologies $\mathbf{T}_{\leq_p}$ and $\mathbf{T}'_{\leq_p}$. It easily follows that they coincide, i.e., $\mathbf{T}_{\leq_p} = \mathbf{T}'_{\leq_p}$. Also, $(X, \mathbf{T}_{\leq_p})$ is a rim-finite dendritic space and $\mathbf{T}_{\leq_p}$ does not depend on the choice of the base-point p.

Suppose that $(X, \mathbf{T})$ is a dendritic space. Then $\mathbf{T}_{\leq_p} \subset \mathbf{T}$ and $\mathbf{T}_{\leq_p} = \mathbf{T}$ if and only if $(X, \mathbf{T})$ is rim-finite. In particular, if $(X, \mathbf{T})$ is a dendron then $\mathbf{T}_{\leq_p}$ coincides with $\mathbf{T}$.

Suppose that $(X, \mathbf{T})$ is a uniquely arc-connected Hausdorff space. Then $\mathbf{T}_{\leq_p}$ coincides with the topology introduced on X by the subbasis $\{Y : Y$ is an arc-component of $X - \{x\}, x \in X\}$ for open sets. It also follows that $(X, \mathbf{T}_{\leq_p})$ is a dendron if and only if $(X, \mathbf{T})$ is nested. The topologies $\mathbf{T}$ and $\mathbf{T}_{\leq_p}$ usually are not comparable, for example, if $(X, \mathbf{T})$ is a nonlocally connected dendroid, then $\mathbf{T}$ and $\mathbf{T}_{\leq_p}$ are different Hausdorff topologies on X. Despite that order-theoretic properties of $(X, \leq_p)$ can be employed to give results concerning $(X, \mathbf{T})$. One of the possible applications is Theorem 4 below. As another example we mention the following fact: If X is a planable dendroid then the set of all ramification points of X can be covered by countably many arcs (see [10, 8.9 and 9.12]).

Let X be a rim-finite dendritic space and $x \in X$. By $r(x)$ we shall denote the *order of ramification* of X at x, i.e., $r(x)$ is the number of components of $X - \{x\}$. We shall say that x is an *end point* (respectively, a *ramification point*) of X if $r(x) = 1$ (respectively, if $r(x) \geq 3$). The following theorem follows from [10, 7.6].

THEOREM 3: Let α be a cardinal number ≥ 3, K be an arc or a half-open arc, s be its end point and $\leq$ be the natural linear ordering of K in which s is the smallest element. Then there exist a rim-finite dendritic space K_α with a point $s_\alpha \in K_\alpha$ and a function $f: K_\alpha \to K$ such that:

(a) $\{s_\alpha\} = f^{-1}(s)$;
(b) if $t \in K$ and t is not the biggest point of $(K, \leq)$, and $x \in f^{-1}(t)$, then $r(x) = \alpha$;
(c) if $x, y \in K_\alpha$ and $x <_{s_\alpha} y$ then $f(x) < f(y)$;
(d) if L is a maximal linearly ordered subset of $(K_\alpha, \leq_{s_\alpha})$ then $f(L) = K$.

The function f as above is said to *strongly fold* K_α onto K. It is never continuous, but $f|_{[s_\alpha, x]}$ is a homeomorphism for each $x \in K_\alpha$. The space K_α has nice uniqueness and universality properties (see [10, 7.7]).

THEOREM 4: [10, 10.2] Let Y be a metrizable hereditarily indecomposable continuum. Then $(C(Y), \mathbf{T}_{\leq Y})$ is homeomorphic to $I_{2^{\aleph_0}}$.

Now, let X be a uniquely arc-connected and I-connected Hausdorff space and $p \in X$. By [10, 8.22], $(X, \mathbf{T}_{\leq p})$ can be embedded into J_α for a sufficiently large cardinal number α. Even if X is nested (e.g., an I-connected dendron), it does not follow that $(X, \mathbf{T}_{\leq p})$ can be embedded into some I_α. Such spaces can be constructed under the assumption that there exists a Souslin line (see [10, 8.26 (ii) or 8.27 (vi)]).

Let X be an $\mathbf{R}$-tree. Thus X is a uniquely arc-connected space which admits a *convex metric* d, that is $d(x,z) = d(x,y) + d(y,z)$ for every $x, z \in X$ and $y \in [x, z]$. Then X is locally arc-connected and it follows that it is a dendritic space. Clearly, only few $\mathbf{R}$-trees can be rim-finite. Therefore, in general, $\mathbf{R}$-trees do not have dendritic compactifications. However, it is possible to embed densely each $\mathbf{R}$-tree into a smooth arboroid [4]. The following result was obtained by Mayer and Oversteegen:

THEOREM 5: [5] The class of $\mathbf{R}$-trees coincides with the class of uniquely arc-connected and locally arc-connected metrizable spaces.

Let X be an $\mathbf{R}$-tree, d be a convex metric on X and $p \in X$. We consider X with its cut-point ordering $\leq_p$. For every $x, y \in X$ there exists the unique $z \in X$ such that $L(z) = L(x) \cap L(y)$, we denote that point by $z = x \wedge y$. We also define $g: X \to [0, \infty[$ by $g(x) = d(p, x)$. Then g *folds* X into $[0, \infty[$, i.e., $g|_{[p,x]}$ is a homeomorphism of $[p, x]$ onto $[0, g(x)]$ for each $x \in X$. Since g is defined with the use of a metric, it is a continuous function. Observe that g and the arc structure of X give enough data to recover the metric d by the rule $d(x,y) = g(x) + g(y) - 2 \cdot g(x \wedge y)$. It also can be shown that the collection $\{m(x): x \in X\} \cup \{g^{-1}([0,t[): t \in]0, \infty[\}$ is a subbasis for open sets in X. In [4] a somewhat reverse procedure was employed to give the sets I_α or $[0, \infty[_\alpha$, $\alpha \geq 3$, a natural $\mathbf{R}$-tree topology, and it was proved that the obtained $\mathbf{R}$-trees have strong universality properties.

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Finite-to-One Mappings on Infinite-dimensional Compacta

ROMAN POL

Institute of Mathematics
University of Warsaw, Banacha 2
02-097 Warszawa, Poland

ABSTRACT: Weakly infinite-dimensional compacta can be classified by means of essential maps onto "transfinite cubes" (Smirnov's compacta). We investigate the behavior of this classification under finite-to-one mappings. In particular, we show that this topic is closely related to an open problem about the invariance of strong infinite-dimensionality under light mappings on compacta. We provide also an analogue for the transfinite dimension ind of Hurewicz's theorem on dimension-rising mappings.

1. INTRODUCTION

We shall consider separable metrizable spaces only, and a compactum is a compact space.

An n-dimensional compactum can be mapped essentially onto the n-cube, but not onto the $(n + 1)$-cube. Compacta which admit an essential map onto the Hilbert cube are called *strongly infinite-dimensional*, and the other ones *weakly infinite-dimensional.*

An outstanding open problem in this topic is the following one [4, Section 2], and Remark 3.2 below.

PROBLEM 1.1: Let $f: X \longrightarrow Y$ be a continuous mapping between compacta such that Y and all the fibers of f are weakly infinite-dimensional. Must X also be weakly infinite-dimensional?

Yu.M. Smirnov [15] defined the compacta $\mathbf{S}_\alpha$, $\alpha < \omega_1$, which display a natural transfinite scale of dimensional complexity between the finite dimension (represented by the euclidean cubes) and the infinity in the strongest form (represented by the Hilbert cube). Subsequently, D.W. Henderson [6] introduced the concept of an essential mapping onto those compacta. For any weakly infinite-dimensional compactum X, the set of ordinals α such that some compactum in X can be mapped essentially onto $\mathbf{S}_\alpha$ is bounded in ω_1 [12], and its supremum (in fact, the maximum) is the Borst-Henderson index $\mathrm{d}(X)$ [1], [2], [4].

We shall show (in Section 3) that the positive answer to Problem 1.1 is equivalent to the statement that the property $\mathrm{d}(X) \geq \alpha$ is preserved by finite-to-one continuous maps, for all α from a closed unbounded set in ω_1.

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To shed more light on the problem, we associate (in Section 4) to each finite-to-one continuous map $f : X \longrightarrow Y$ between compacta a countable ordinal ord f, the transfinite order of f. Then we show (in Section 5) that the property $\mathrm{d}(X) \geq \alpha$ is preserved by continuous maps with ord $f < \alpha$, for all α from a closed unbounded set in ω_1.

Our proofs do not indicate, however, what happens for concrete ordinals α.

In the last section the transfinite order ord f is used to provide an analogue of Hurewicz's dimension-raising theorem [8, p. 93] for the transfinite dimension ind. As before, our approach gives no exact formula.

2. SMIRNOV'S COMPACTA, ESSENTIAL MAPS, AND THE BORST-HENDERSON INDEX d(X)

In this section we introduce the main notions considered in this paper. Our terminology follows [9] and [11].

2.1. THE HYPERSPACES AND THE FUNCTION SPACES: The hyperspace $\mathcal{K}(X)$ of a compactum X is the space of nonempty compact subsets of X with the Vietoris topology [9, Section 17]. Given compacta X, Y we denote by $C(X, Y)$ the space of continuous functions from X to Y with the topology of uniform convergence.

2.2. SMIRNOV'S COMPACTA: Yu.M. Smirnov [15] defined for each countable ordinal α a compactum $\mathbf{S}_\alpha$ in the following way: $\mathbf{S}_n = \mathbf{I}^n$ is the euclidean n-cube, $\mathbf{S}_{\alpha+1} = \mathbf{S}_\alpha \times \mathbf{I}$ and, for limit α, $\mathbf{S}_\alpha$ is the one-point compactification of the free union of $\mathbf{S}_\beta$ with $\beta < \alpha$. Every compactum $\mathbf{S}_\alpha$ has only countably many components, , each being a finite-dimensional cube.

2.3. ESSENTIAL MAPS, WEAKLY INFINITE-DIMENSIONAL COMPACTA AND THE INDEX d(X): A continuous mapping $f : X \longrightarrow \mathbf{I}^n$ onto the n-cube is essential if for any continuous map $g : X \longrightarrow \mathbf{I}^n$ which coincides with f on the preimage $f^{-1}(\partial\mathbf{I}^n)$ of the boundary, $g(X) = \mathbf{I}^n$. We say in this case that f covers the cube essentially.

A continuous mapping $f : X \longrightarrow \mathbf{I}^\infty$ from a compactum X onto the Hilbert cube is essential if for each projection $p_n : \mathbf{I}^\infty \longrightarrow \mathbf{I}^n$ onto the n-th coordinate product, the composition $p_n \circ f : X \longrightarrow \mathbf{I}^n$ is essential.

A compactum X is called *weakly infinite-dimensional* if X can not be mapped essentially onto $\mathbf{I}^\infty$, or *strongly infinite-dimensional*, if such a map exists on X.

Following D.W. Henderson [6] we shall say that a continuous mapping $f : X \longrightarrow \mathbf{S}_\alpha$ from a compactum X onto Smirnov's compactum $\mathbf{S}_\alpha$ is *essential* if f covers essentially each open component C of $\mathbf{S}_\alpha$, i.e., f maps essentially $f^{-1}(C)$ onto the cube C.

We changed slightly the original definition of Henderson, who considered $\mathbf{S}_\alpha$ embedded canonically into AR-compacta $\mathbf{J}_\alpha$, defined along with $\mathbf{S}_\alpha$, but by Proposition 3 in [6], this is an equivalent approach.

P. Borst [1], [2] made a significant progress in understanding this notion. A comprehensive account of the topic is given by V.A. Chatyrko [4].

Let us associate to each compactum X its Borst-Henderson index d(X) (denoted in [2, 4.3] by Ess(X)) : d(X) is the supremum of the ordinals α such that some compactum in X maps essentially onto $\mathbf{S}_\alpha$, if the supremum is countable, and $d(X) = \infty$, otherwise.

Below we list some basic properties of the index.

2.3.1: If $\mathrm{d}(X)$ is finite then $\mathrm{d}(X) = \dim X$, and $\mathrm{d}(\mathbf{S}_\alpha) = \alpha$, [6], [15, Theorems 3 and 5].

2.3.2: For any compactum X, $d(X) = \infty$ if and only if X is strongly infinite-dimensional [12, Theorem 2.1] and [2, Theorem 4.3.1].

2.3.3: By [2, Theorem 4.3.2], if $\mathrm{d}(X) \neq \infty$, then the supremum in the definition of $\mathrm{d}(X)$ is attained, i.e., some compactum in X maps essentially onto $\mathbf{S}_{\mathrm{d}(X)}$.

2.3.4: If $\mathrm{d}(X) \geq \alpha + 1$ then there exists a pair of disjoint closed sets A, B in X such that for each partition S in X between A and B, $\mathrm{d}(S) \geq \alpha$, [6, Proposition 4] and [2, 4.3].

2.3.5: For each $\xi < \omega_1$ there exists a weakly infinite-dimensional compactum Z which contains topologically all compacta X with $\mathrm{d}(X) \leq \xi$ [12, 5.1] and [1, Theorem 3.3.8].

2.3.6: For any analytic set S in the hyperspace of the Hilbert cube $\mathcal{K}(\mathbf{I}^\infty)$ the elements of which are weakly infinite-dimensional compacta, $\sup\{\mathrm{d}(X) : X \in S\} < \omega_1$ [12], [1, Theorem 3.3.8].

2.4: THE LENGTH OF A WELL-FOUNDED RELATION: Let $\prec$ be a partial order on a set Λ. The order is well-founded on $\Gamma \subset \Lambda$ if there is no infinite descending sequence $a_1 \succ a_2 \succ \ldots$ in Γ. The length of $(\Gamma, \prec)$ is then defined inductively as follows [10, 2.*D*].

For $u \in \Gamma$, $\mathrm{rank}_\Gamma u = 1$ means that there is no $v \in \Gamma$ with $v \prec u$,

$$\mathrm{rank}_\Gamma u = \sup\{\mathrm{rank}_\Gamma v + 1 : v \in \Gamma, v \prec u\}$$

and, finally,

$$\mathrm{length}\Gamma = \sup\{\mathrm{rank}_\Gamma u : u \in \Gamma\}.$$

If $\prec$ is not well-founded on Γ, we let $\mathrm{length}\Gamma = \infty$.

Let $\prec$ be a partial order on a countable set Λ. We denote by $\mathbf{2}^\Lambda$ the space of all subsets of Λ, identified with the characteristic functions, equipped with the pointwise topology. For each $\xi < \omega_1$, the set

$$\{\Gamma \subset \Lambda : \mathrm{length}\Gamma \leq \xi\} \text{ is Borel.}$$

This can be easily verified by induction [9, Section 38, IX].

3. FINITE-TO-ONE MAPPINGS AND THE INDEX d(X)

In this section we shall show that Problem 1.1 can be translated into a problem concerning the behavior of the Borst-Henderson index $\mathrm{d}(X)$ (see Section 2.3) under finite-to-one mappings. A theorem of Toruńczyk [16] will play a key role in our reasoning.

THEOREM 3.1: The positive answer to Problem 1.1 is equivalent to the following statement: for all α from a closed unbounded set in ω_1, if $f : X \longrightarrow Y$ is a finite-to-one continuous mapping between compacta and $d(X) \geq \alpha$, then $d(Y) \geq \alpha$.

Let us notice that if $f : X \longrightarrow Y$ is as in the statement and $d(X)$ is finite, then $d(Y) \geq d(X)$ by a theorem of Hurewicz on dimension — lowering maps [8, Section 4, 2.3.1] and if $d(X) = \infty$, then $d(Y) = \infty$ [14, 2.3.2].

REMARK 3.2: A continuous mapping $f : X \longrightarrow Y$ between compacta is *light* if the fibers of f are zero-dimensional [7]. Let us notice that Problem 1.1 is equivalent to the problem of whether light mappings preserve strong infinite-dimensionality. Indeed, suppose $f : X \longrightarrow Y$ is a continuous map between compacta such that X is strongly infinite-dimensional but Y and all the fibers of f are weakly infinite-dimensional. By a theorem of Henderson [5], X contains a strongly infinite-dimensional compactum K, each weakly infinite-dimensional subcompactum of which is at most zero-dimensional. Then f restricted to K is light.

3.3. *Proof of the implication* $\Leftarrow$*:* Assume that Problem 1.1 has a negative answer, i.e., by Remark 3.2, there exists a light mapping $f : K \longrightarrow L$ of a strongly infinite-dimensional compactum K onto a weakly infinite-dimensional compactum L. We have to show that the statement in Theorem 3.1 is in this case false.

We shall consider K and L with fixed metrics, and we shall call a continuous map on K or L an ε-map, provided that its fibers have diameters less than ε.

Since f is light, for each natural n there exists $\varepsilon_n > 0$ such that f maps continua in K of diameter $\geq 1/n$ onto continua in L of diameter $\geq \varepsilon_n$.

Let $l_n : L \longrightarrow L_n$ be an ε_n-map onto a finite polytope, and let $l_n \circ f = u_n \circ k_n$, $k_n : K \longrightarrow K_n$, $u_n : K_n \longrightarrow L_n$, be the monotone-light factorization of $l_n \circ f$, i.e., u_n is light and the fibers of k_n are connected [7, Section 3–7]. The continua $k_n^{-1}(s)$ are taken by $l_n \circ f$ to points, hence $\operatorname{diam} f(k_n^{-1}(s)) < \varepsilon_n$, and therefore k_n is a $1/n$-map. Since u_n is light, K_n is finite-dimensional.

We now shall repeat a reasoning from [12, Section 3]. Let $\mathbf{2}^\infty$ be the Cantor set and let $Q = \{q_1, q_2, \ldots\}$ be a countable set dense in $\mathbf{2}^\infty$. In the product $\mathbf{2}^\infty \times K$ we attach to each section $\{q_n\} \times K$ the compactum K_n by the map k_n, and in the product $\mathbf{2}^\infty \times L$ we attach to each section $\{q_n\} \times L$ the polytope L_n by the map l_n. In effect, we get compacta K^* and L^*, respectively. The projections from $\mathbf{2}^\infty \times K$ and $\mathbf{2}^\infty \times L$ onto the first axis induce continuous mappings $p_K : K^* \longrightarrow \mathbf{2}^\infty$ and $p_L : L^* \longrightarrow \mathbf{2}^\infty$, and the map $id \times f : \mathbf{2}^\infty \times K \longrightarrow \mathbf{2}^\infty \times L$ induces a light mapping $f^* : K^* \longrightarrow L^*$ such that $p_L \circ f^* = p_K$.

Let C be the collection of nonempty compact subsets of Q. For each $C \in C$ we set $K(C) = p_K^{-1}(C)$ and $L(C) = p_L^{-1}(C)$. Then f^* maps $K(C)$ onto $L(C)$. Since K is strongly infinite-dimensional, by [12, Section 2, Proposition 3.1] and [1, 3.3], $\sup\{d(K(C)) : C \in C\} = \omega_1$. On the other hand, L is weakly infinite-dimensional, and so is L^*, and therefore $\xi = d(L^*) < \omega_1$.

Let $\alpha > \xi$ and let us consider $C \in C$ with $d(K(C)) \geq \alpha + 1$. For each $q \in C$, f^* is a light map from $K_q = p_K^{-1}(q)$ onto $L_q = p_L^{-1}(q)$. Since L_q is finite-dimensional, a theorem of Toruńczyk [16] provides a zero-dimensional set $N_q \subset K_q$ such that f^* restricted to $K_q \backslash N_q$ is finite-to-one. Let A and B be a pair of disjoint closed sets in $K(C)$ described in Section 2.3.4 (where $X = K(C)$), and let S be a partition in $K(C)$ between A and B, disjoint from the zero-dimensional set $\bigcup\{N_q: q \in C\}$. Then $\mathrm{d}(S) \geq \alpha$, f^* restricted to S is finite-to-one, and $\mathrm{d}(f^*(S)) \leq \xi$. We conclude that the statement in Theorem 3.2 is false for all $\alpha > \xi$. ❑

3.4. *Proof of the implication* $\Rightarrow$: Assume that Problem 1.1 has a positive answer. Let us fix $\xi < \omega_1$. We shall find $\varphi(\xi) < \omega_1$ such that for any light mapping $f : K \longrightarrow L$ between compacta, if $\mathrm{d}(K) > \varphi(\xi)$ then $\mathrm{d}(L) > \xi$. Then the closed unbounded set of countable limit ordinals α with $\varphi(\beta) < \alpha$ for all $\beta < \alpha$ will witness the validity of the statement in Theorem 3.1.

To this end, let $Z \subset \mathbf{I}^\infty$ be a weakly infinite-dimensional compactum described in Section 2.3.5, and let (cf. Section 2.1),

$$\mathcal{A} = \{(K, u) \in \mathcal{K}(\mathbf{I}^\infty) \times C(\mathbf{I}^\infty, \mathbf{I}^\infty) : u(K) \subset Z \text{ and } u \text{ restricted to } K \text{ is light}\}.$$

The set $\mathcal{A}$ is of type G_δ, and its projection on the first axis,

$$\operatorname{proj} \mathcal{A} \subset \mathcal{K}(\mathbf{I}^\infty) \text{ is analytic.}$$

By our assumption, each $K \in \mathcal{A}$ is weakly infinite-dimensional, and by Section 2.3.6,

$$\varphi(\xi) = \sup\{\mathrm{d}(K) : K \in \operatorname{proj} \mathcal{A}\} < \omega_1.$$

Let $f : K \longrightarrow L$ be a light mapping between compacta with $\mathrm{d}(K) > \varphi(\xi)$. Aiming at a contradiction, suppose that $\mathrm{d}(L) \leq \xi$. We can assume then that $L \subset Z$, $K \subset \mathbf{I}^\infty$, and let $f^* \in C(\mathbf{I}^\infty, \mathbf{I}^\infty)$ be a continuous extension of f. The pair (K, f^*) belongs to $\mathcal{A}$, hence $K \in \operatorname{proj} \mathcal{A}$, but this contradics the definition of $\varphi(\xi)$.

3.5. REMARK: The reasoning in Section 3.3 shows that Theorem 3.1 remains true if the statement is restricted to compacta X, Y which have an upper semicontinuous decomposition into countably many finite-dimensional compacta (such compacta can be embedded in one of Smirnov's compacta [12, Lemma 2.1]).

4. THE TRANSFINITE ORDER OF FINITE-TO-ONE MAPPINGS ON COMPACTA

Let X be a compactum. A *regular partition* of X is a finite cover $\mathcal{A}$ of X by nonempty regularly closed sets (i.e., sets which are the closures of their interiors) with pairwise disjoint interiors. We shall consider the collection $\Gamma(X)$ of all regular partitions of X with the partial order $\prec$, where $\mathcal{A} \prec \mathcal{B}$ means that $\mathcal{A}$ refines $\mathcal{B}$ and there exist $A \in \mathcal{A}$, $B \in \mathcal{B}$ such that $A \neq B$ and $A \subset \operatorname{int} B$.

Let $f : X \longrightarrow Y$ be a continuous mapping between compacta. We set

$$\Gamma(f) = \{\mathcal{A} \in \Gamma(X) : \bigcap\{f(\operatorname{int} A) : A \in \mathcal{A}\} \neq \varnothing\}. \tag{1}$$

In the sequel we shall use terminology introduced in Section 2.4.

LEMMA 4.1: The order $\prec$ is well-founded on $\Gamma(f)$ if and only if f is finite-to-one.

Proof: Assume that $\prec$ is not well-founded on $\Gamma(f)$, and let $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \ldots$ be an infinite descending sequence in $\Gamma(f)$. We shall choose numbers $n(1) < n(2) < \ldots$ and disjoint families $\mathcal{D}_i \subset \mathcal{A}_{n(i)}$ such that $\mathcal{D}_i$ has at least i elements and

$$\bigcap\{f(A): A \in \mathcal{D}_{i+1}\} \subset \bigcap\{f(A): A \in \mathcal{D}_i\}. \tag{2}$$

Suppose $n(i)$ and $\mathcal{D}_i$ have been chosen. There exists $E \in \mathcal{A}_{n(i)}$ such that for infinitely many $j > n(i)$ the relation $\mathcal{A}_{j+1} \prec \mathcal{A}_j$ is witnessed by some elements contained in E. Then we can find $k > n(i)$, $A \in \mathcal{A}_k$ with $A \neq E$ and $A \subset \operatorname{int}E$, and $l > k$, $B \in \mathcal{A}_l$, $C \in \mathcal{A}_k$ with $B \neq C$ and $B \subset \operatorname{int}C \subset E$. We set $n(i+1) = l$ and define $\mathcal{D}_{i+1}$ as follows. First, each $D \in \mathcal{D}_i \backslash \{E\}$ is replaced by an element of $\mathcal{A}_l$ contained in D. If $E \notin \mathcal{D}_i$ we then add B. Assume that $E \in \mathcal{D}_i$. Then E is replaced by B and an element of $\mathcal{A}_l$ contained either in A or in $E \backslash \operatorname{int}A$, depending on whether $A \neq C$ or $A = C$.

Now, by compactness, we get from (2) a point $y \in \bigcap_i(\bigcap\{f(A): A \in \mathcal{D}_i\})$. The fiber $f^{-1}(y)$ intersects each element of every family $\mathcal{D}_i$, and hence it is infinite.

On the other hand, assume that f has an infinite fiber $f^{-1}(y)$ and let $A_1, A_2, \ldots$ be pairwise disjoint regularly closed sets in X with $\operatorname{int}A_i \cap f^{-1}(y) \neq \varnothing$. Then each family $\mathcal{A}_i = \{A_1, \ldots, A_i, X \backslash(\operatorname{int}A_1 \cup \ldots \cup \operatorname{int}A_i)\}$ belongs to $\Gamma(f)$ and $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \ldots$. ❑

DEFINITION 4.2: The *transfinite order* of the mapping f is now defined as follows: $\operatorname{ord} f = \operatorname{length} \Gamma(f)$, if f is finite-to-one, and $\operatorname{ord} f = \infty$, otherwise.

Since, apart from some trivial cases, $\Gamma(f)$ is uncountable, it is not immediately clear that $\operatorname{ord} f \neq \infty$ is a countable ordinal.

LEMMA 4.3: If f is finite-to-one, then $\operatorname{ord} f$ is a countable ordinal.

Proof: Let $\mathcal{K}(X)$ be the hyperspace of the domain of f (see Section 2.1) and let E be the set of points $(A_1, \ldots, A_n, \varnothing, \varnothing, \ldots)$ in the countable product $\mathcal{K}(X) \times \mathcal{K}(X) \times \ldots$, such that $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Gamma(f)$. Let $\prec^*$ be the partial order on E corresponding to the order $\prec$ on $\Gamma(f)$. Then both orders are well-founded and have the same length. Using the Borel measurability of the map $K \longrightarrow \operatorname{cl}(X \backslash K)$ from $\mathcal{K}(X)$ into itself [9, Section VIII], one readily checks that $\prec^*$ is an analytic order. The assertion is now a consequence of the theorem that analytic well-founded relations have countable length (see [10, 2G]).

REMARK 4.4: Finite order $\operatorname{ord} f$ is the maximal cardinality of the fibers of f. If $f: 2^\infty \longrightarrow \mathbf{S}_\alpha$ is a continuous mapping from the Cantor set onto Smirnov's compactum $\mathbf{S}_\alpha$, then $\operatorname{ord} f \geq \alpha$ [13, Proposition 4.1] (the proof shows that ind can be replaced in this proposition by Ind) and [15, Theorem 5].

5. THE TRANSFINITE ORDER OF MAPS AND THE INDEX d(X)

The following result provides some information concerning the statement in Theorem 3.1.

THEOREM 5.1: For all α from a closed unbounded set in ω_1, if $f: X \longrightarrow Y$ is a continuous map between compacta with $\operatorname{ord} f < \alpha$ and $\mathrm{d}(X) \geq \alpha$, then $\mathrm{d}(Y) \geq \alpha$.

Proof: We shall check the following:
Claim: For each $\xi < \omega_1$ there exists $\varphi(\xi) < \omega_1$ such that if $f : X \longrightarrow Y$ is a continuous map between compacta with $\operatorname{ord} f \le \xi$ and $d(Y) \le \xi$, then $d(X) \le \varphi(\xi)$.

Once the claim is established, the assertion will follow instantly. Indeed, let Σ be a closed unbounded set of limit ordinals in ω_1 such that if $\beta < \alpha$ and $\alpha \in \Sigma$ then $\varphi(\beta) < \alpha$. Let $f : X \longrightarrow Y$ satisfy $\operatorname{ord} f < \alpha$ and $d(X) \ge \alpha$, for some $\alpha \in \Sigma$. If ξ is strictly between $\operatorname{ord} f$ and α, then $d(X) > \varphi(\xi)$, and by the claim, $d(Y) > \xi$. Therefore, $d(Y) \ge \alpha$.

To prove the claim, let us fix $\xi < \omega_1$, and let $Z \subset \mathbf{I}^\infty$ be the weakly infinite-dimensional compactum described in Section 2.3.5.

We shall call a set A in the Hilbert cube $\mathbf{I}^\infty$ *elementary*, if A is a finite union of cubes $[a_1, b_1] \times \ldots \times [a_n, b_n] \times \mathbf{I} \times \ldots$ with rational $a_i < b_i$, and we shall say that a regular partition of $\mathbf{I}^\infty$ is *elementary* if it consists of elementary sets. We denote by Λ the countable collection of all elementary partitions of $\mathbf{I}^\infty$.

For any pair (see Section 2.1)

$$(X, u) \in \mathcal{K}(\mathbf{I}^\infty) \times C(\mathbf{I}^\infty, \mathbf{I}^\infty),$$

we set (cf. Section 4, formula (1)),

$$\Lambda(X, u) = \{\mathcal{A} \in \Lambda : \bigcap\{u(\operatorname{int} A \cap X) : A \in \mathcal{A}\} \neq \emptyset\}.$$

As in the proof of Lemma 4.1, one checks that u restricted to X is finite-to-one if and only if the order $\prec$ is well-founded on $\Lambda(X, u)$. Let 2^Λ be the space of all subsets of Λ with the pointwise topology. Since the map $(X, u) \longrightarrow \Lambda(X, u)$ from $\mathcal{K}(\mathbf{I}^\infty) \times C(\mathbf{I}^\infty, \mathbf{I}^\infty)$ to 2^Λ is Borel, by the remark at the end of 2.4, the set

$$\mathcal{B} = \{(X, u) : \operatorname{length} \Lambda(X, u) \le \xi\} \text{ is Borel,} \tag{1}$$

and also the set

$$\mathcal{E} = \{(X, u) \in \mathcal{B} : u(X) \subset Z\} \text{ is Borel.}$$

Therefore, the projection parallel to the second axis,

$$\operatorname{proj} \mathcal{E} \subset \mathcal{K}(\mathbf{I}^\infty) \text{ is analytic.}$$

Let $X \in \operatorname{proj} \mathcal{E}$, and let $(X, u) \in \mathcal{B}$. Then $\prec$ is well-founded on $\Lambda(X, u)$, and hence u restricted to X is finite-to-one. Since $u(X) \subset Z$ and Z is weakly infinite-dimensional, we conclude that so is X [14]. By Section 2.3.6,

$$\varphi(\xi) = \sup\{d(X) : X \in \operatorname{proj} \mathcal{E}\} < \omega_1. \tag{2}$$

Let us check that $\varphi(\xi)$ has the required property. Let $f : X \longrightarrow Y$ be a continuous mapping between compacta, with $\operatorname{ord} f \le \xi$ and $d(Y) \le \xi$. We can assume that $Y \subset Z$. We can also assume that X is embedded in $\mathbf{I}^\infty$ in such a way that for each elementary set A in $\mathbf{I}^\infty$, the intersection of the boundary $\operatorname{bd} A$ with X has empty interior in the space X. Indeed, if we fix a countable set D dense in X, then for each elementary set A and $d \in D$, the set $G(d, A)$ of embeddings h of X in $\mathbf{I}^\infty$ with $h(d) \notin \operatorname{bd} A$ is residual in $C(X, \mathbf{I}^\infty)$, and any $h \in \bigcap\{G(d, A) : d \in D, A \text{ — elementary}\}$ is an appropriate embedding. Let $f^* : \mathbf{I}^\infty \longrightarrow \mathbf{I}^\infty$ be a continuous extension of f. The traces on X of elementary sets in $\mathbf{I}^\infty$ are regularly closed in X, which provides an order $\prec$-preserving embedding of $\Lambda(X, f^*)$ into $\Gamma(f)$ (cf. Section 4, (1)). Therefore, length $\Lambda(X, f^*) \leq$ length $\Gamma(f) = \text{ord} f \leq \xi$, hence $(X, f^*) \in \mathcal{E}$ and $X \in \text{proj}\mathcal{E}$. By (2), $d(X) \leq \varphi(\xi)$, which completes the proof. ❑

6. FINITE-TO-ONE MAPPINGS AND THE TRANSFINITE DIMENSION ind

A space is countable-dimensional if it is a countable union of zero-dimensional sets. Let ind be the extension by transfinite induction of the classical Menger-Urysohn dimension [8, IV.6.B], and [11, Definition VI.1]. For each completely metrizable space X, ind$X$ is defined if and only if X is countable-dimensional, and then ind$X < \omega_1$.

Hurewicz's theorem on dimension-raising mappings [8, Remark, p.93], [11, II.6], asserts that if $f: X \longrightarrow Y$ is a continuous mapping of finite order on a finite-dimensional compactum, then

$$\text{ind} f(X) \leq \text{ind} X + \text{ord} f - 1.$$

We shall show that certain analogue of this theorem can also be obtained in the transfinite case.

THEOREM 6.1: There exists a function $\lambda: \omega_1 \times \omega_1 \longrightarrow \omega_1$ such that for any continuous finite-to-one mapping $f: X \longrightarrow Y$ on a countable-dimensional compactum X,

$$\text{ind} f(X) \leq \lambda(\text{ind} X, \text{ord} f).$$

Proof: Let us fix countable ordinals η and ξ. By [12, Theorem 1.1] there exists a continuous function $\Phi: \omega^\infty \longrightarrow \mathcal{K}(\mathbf{I}^\infty)$ from the irrationals to the hyperspace of the Hilbert cube such that for each $X \in \mathcal{K}(\mathbf{I}^\infty)$ with ind$X \leq \eta$ there is $z \in \omega^\infty$ with $X = \Phi(z)$, and

$$\text{ind}\{(z, x) \in \omega^\infty \times \mathcal{K}(\mathbf{I}^\infty) : x \in \Phi(z)\} = \eta. \tag{1}$$

Let $\mathcal{B}$ be the set defined in the proof of Theorem 5.1 by formula (1). Then the set

$$\mathcal{F} = \{(z, u) \in \omega^\infty \times C(\mathbf{I}^\infty, \mathbf{I}^\infty) : (\Phi(z), u) \in \mathcal{B}\} \tag{2}$$

is Borel. Let $s \rightarrow (v(s), w(s))$ be a continuous map from ω^∞ onto $\mathcal{F}$. The set

$$F = \{(s, x) \in \omega^\infty \times \mathbf{I}^\infty : x \in \Phi(v(s))\} \tag{3}$$

is closed in $\omega^\infty \times \mathbf{I}^\infty$, and the mapping

$$k: F \longrightarrow \omega^\infty \times \mathbf{I}^\infty, \quad k(s, x) = (s, w(s)(x)) \tag{4}$$

is perfect, the projection parallel to $\mathbf{I}^\infty$ restricted to F being pefect [3, Section 10]. As was stated in the proof of Theorem 5.1, if $(X, u) \in \mathcal{B}$ then u is finite-to-one on

X, hence by (2) and (3), $w(s)$ is finite-to-one on the vertical section of F at s, and in effect, k is finite-to-one.

From [12, 3.2] (where $M = F$, $\tilde{M}$ is defined by (1), and $t = v$) we infer that indF $\leq \eta$, and hence F is countable-dimensional. Because finite-to-one perfect maps preserve countable-dimensionality [11, Theorem VI.6], $k(F)$ is a completely metrizable countable-dimensional space, and hence

$$\lambda(\eta, \xi) = \mathrm{ind}k(F) < \omega_1. \tag{5}$$

To end the proof, let us consider a continuous mapping $f : X \longrightarrow Y$ of a compactum X onto Y with ord $f \leq \xi$ and ind$X \leq \eta$. Let $Y \subset \mathbf{I}^\infty$ and let (X, f^*) be as at the end of the proof of Theorem 5.1, in particular $(X, f^*) \in \mathcal{B}$. For $z \in \omega^\infty$ such that $X = \Phi(z)$ we have then $(z, f^*) \in \mathcal{F}$, and hence $(z, f^*) = (v(s), w(s))$ for some $s \in \omega^\infty$. We conclude that $Y = f(X) = w(s)(X)$, is the vertical section of $k(F)$ at s (see (4)), and by (5), ind$Y \leq \lambda(\eta, \xi)$. ❑

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Continuous Selections of Starlike-valued Mappings

P.V. SEMENOV

Department of Mathematics
Moscow State Pedagogical University
Moscow, Russia

A nonempty subset M of a vector space V is said to be starlike if M can be represented as a union $\cup M_\gamma$, $\gamma \in \Gamma$, of a centered family $\{M_\gamma\}$ of a convex subset $M_\gamma \subset V$. The purpose of this note is to find a condition for the existence of continuous single-valued selections of arbitrary lower semicontinuous starlike-valued mappings from a paracompact into a Hilbert space. In other words, we try to generalize a well-known selection theorem of E. Michael [1] for lower semicontinuous convex-valued mappings in the direction of eliminating the convexity condition. The main technical point of the proof is another selection theorem of E. Michael for paraconvex-valued mappings [2]. For simplicity, we consider the case of the Hilbert space as the range of lower semicontinuous mappings. In a joint article with D. Repovš [3], some other applications of selection theorem for paraconvex-valued mappings were given.

In Section 1, we give an example which shows that, in general, the answer for the question above is negative, even for mappings with values equal to the union of two segments. Such an example is an analogue of the $\sin(1/x)$-example (see [1]). In Section 2, we give some sufficient conditions for an affirmative answer. To formulate these conditions, we need the following preliminary definitions.

DEFINITION 1: Let $\{M_\gamma\}$, $\gamma \in \Gamma$ be a family of sets. For each subset E of the union $\bigcup M_\gamma$, the set $\{\gamma \in \Gamma : E \cap M_\gamma \neq \varnothing\}$ is said to be the support of E and is denoted by $\mathrm{supp}(E)$.

DEFINITION 2: Let $\{M_\gamma\}$, $\gamma \in \Gamma$ be a family of sets. A subset E of the union $\bigcup M_\gamma$ is said to be an exact subset if E is finite and for every $\gamma \in \mathrm{supp}(E)$, the intersection $E \cap M_\gamma$ consists of a single point.

DEFINITION 3: Let $\beta \in [0, \pi]$. A closed nonempty subset M of a Hilbert space H is said to be β-starlike if M can be represented as a union $\bigcup M_\gamma$, $\gamma \in \Gamma$, of nonempty convex subsets $M_\gamma \subset H$ such that for each exact subset $E \subset \bigcup M_\gamma = M$, there exists $y \in \bigcap M_\gamma$, $\gamma \in \mathrm{supp}(E)$, where the angle $\angle xyz$ is more than or equal to β for all $x \in E$, $z \in E$, $x \neq z$.

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THEOREM 4: Let $\beta \in (0, \pi]$. Then any lower semicontinuous β-starlike-valued mapping from a paracompact into a Hilbert space admits a single-valued continuous selection.

A typical example of a β-starlike set is a β-bouquet M centered at the point y, i.e., the union of convex sets M_γ such that $M_\gamma \cap M_\lambda = \{y\}$, and $\angle xyz \geq \beta$, for all $x \in M_\gamma \backslash \{y\}$, $z \in M_\lambda \backslash \{y\}$, $\gamma \neq \lambda$. We assume that Definition 3 automatically holds when an exact subset E consists of a single point $\{y\}$: in this case there are no angles $\angle xyz$ with $x \neq z$. Hence, every closed nonempty convex subset $M \subset H$ is a π-starlike subset of H. Indeed, we can put $M = M \cup M$ and then each exact subset $E \subset M$ consists of a single point. Thus, Theorem 4 is a generalization of the selection theorem of E. Michael for convex-valued mappings.

EXAMPLE 1: There exists a lower semicontinuous (in fact, continuous) mapping F: $[0, 1] \rightarrow \mathbb{R}^2$ such that all sets $F(t)$ with $t \in [0, 1]$ are unions of two intersecting segments which admit no single-valued continuous selections.

Construction

Let $\{a_n\}$ be an arbitrary monotone decreasing sequence of positive numbers, tending to zero and let $a_1 = 1$. Let $b_n = (a_n + a_{n+1})/2$, $c_n = (a_{n+1} + b_n)/2$, $d_n = (b_n + a_n)/2$. We put:

(a) $F(a_n) = [(b_n, 1), (a_n, -1)]$;
(b) $F(d_n) = [(a_{n+1}, -1), (b_n, 1)] \cup F(a_n)$;
(c) $F(b_n) = [(a_{n+1}, -1), (b_n, 1)]$;
(d) $F(c_n) = [(b_{n+1}, 1), (a_{n+1}, -1) \cup F(b_n)$; and
(e) $F(0) = [(0, 1), (0, 1)]$.

When the parameter t tends uniformily from a_n to d_n, $F(t)$ is equal to the union of the segment $F(a_n)$ and the uniformly growing (from zero to $F(b_n)$) segment. When the parameter t tends uniformily from d_n to b_n, $F(t)$ is equal to the union of the segment $F(b_n)$ and the uniformly shrinking (from $F(a_n)$ to zero) segment. For $t \in [a_{n+1}, b_n]$, the values for $F(t)$ can be defined in an analogous manner.

In this example the set of all angles between segments which form the values $F(t)$ has no positive infimum. Theorem 4 states that the existence of positive infimum for all plane angels in this situation guarantees the existence of continuous selections.

We shall use some (equivalent) modification of the original notion of α-paraconvexity, given in [2].

DEFINITION 5: Let $\alpha \in \mathbb{R}$. A nonempty closed subset P of a Banach space B is called α-paraconvex if for each $n \in \mathbb{N}$, each point $x_1, x_2, \ldots, x_n \in P$ and for each point $q \in [x_1, x_2, \ldots, x_n]$ the distance $\operatorname{dist}(q, P)$ is less than or equal to $\alpha \cdot \operatorname{Rad}\{x_1, x_2, \ldots, x_n\}$.

In this definition, $[x_1, x_2, \ldots, x_n]$ denotes the convex hull of the points $x_1, x_2, \ldots, x_n$ and $\operatorname{Rad}\{x_1, x_2, \ldots, x_n\}$ denotes the infimum of the radii of all closed balls which contains all points $x_1, x_2, \ldots, x_n$. In the case of the Hilbert space there exists a unique closed ball with the radius $\mathrm{Rad}\{x_1, x_2, \ldots, x_n\}$ which contains all points $x_1, x_2, \ldots, x_n$; we denote the center of this ball $c(x_1, x_2, \ldots, x_n)$.

THEOREM 6: [2] Let $\alpha \in [0, 1]$. Then every lower semicontinuous α-paraconvex-valued mapping from a paracompact into Banach space admits a single-valued continuous selection.

Now we pass to the proof of Theorem 4. Our first lemma states that in the Hilbert space for α-paraconvexity of a set P, it suffices to control only the distances $\mathrm{dist}(c(x_1, x_2, \ldots, x_n), P)$ for $x_1, x_2, \ldots, x_n$ from the set P.

LEMMA 7:[a] Let for each $n \in \mathbb{N}$ and for each $x_1, x_2, \ldots, x_n$ from a closed subset P of a Hilbert space H the following inequality holds

$$\mathrm{dist}(c(x_1, x_2, \ldots, x_n), P) \leq \alpha \cdot \mathrm{Rad}\{x_1, x_2, \ldots, x_n\}.$$

Then the set P is an α_1-paraconvex subset of H, where $(1 - \alpha_1)$ is the positive root of the equation

$$\alpha + (2t - t^2)^{1/2} = 1 - t.$$

The following lemma shows the significance of the notion of exact subset.

LEMMA 8:[a] Let V be a vector space and let M_γ, $\gamma \in \Gamma$, be a convex subset of V. Then

$$\mathrm{conv}(\bigcup M_\gamma) = \bigcup\{\mathrm{conv}(E) : E \text{ is an exact subset of } \bigcup M_\gamma\}.$$

The following purely geometric lemma is the main technical point in the proof of Theorem 4.

LEMMA 9:[a] Let $y, x_1, x_2, \ldots, x_n$ be points in H such that for some $\beta \in (0, \pi]$ and for all $i \neq j$, the following inequalities hold:

$$\angle x_i y x_j \geq \beta.$$

Then there exists $k \in \{1, 2, \ldots, n\}$ such that

$$\angle y x_k c \leq (\pi - \beta)/2$$

where $c = c(x_1, x_2, \ldots, x_n)$.

After Lemmas 7, 8, and 9, we can show that β-starlike subsets of a Hilbert space H, $\beta \in (0, \pi]$, are in fact, α-paraconvex subsets of H, for some $\alpha = \alpha(\beta) \in [0,1)$.

Let $M = \bigcup M_\gamma$, $\gamma \in \Gamma$ be the needed representation of β-starlike set M (see Definition 3), and let $q \in \mathrm{conv}(M)$. By Lemma 8, there exists an exact subset $E = \{x_1, x_2, \ldots, x_n\} \subset M$ such that $q \in [x_1, x_2, \ldots, x_n]$. So, by Definition 3, we can find $y \in \bigcap M_\gamma$, $\gamma \in \mathrm{supp}(E)$, such that $\angle x_i y x_j \geq \beta$ for all $i \neq j$. By convexity of M_γ and by Lemma 9, we obtain that $\mathrm{dist}(c, M) \leq \alpha \cdot R$, where $c(x_1, x_2, \ldots, x_n)$, $R = \mathrm{Rad}(x_1, x_2, \ldots, x_n)$, $\alpha = \sin((\pi - \beta)/2)$. So, by Lemma 7, we obtain the α_1-paraconvexity of M, for some $\alpha_1 \in [0, 1)$ and an applicaton of Theorem 6 completes the proof of Theorem 4. ❑

[a]Proof in: 1996. Sib. Mat. Journal. **37**: 399–405.

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On Free Topological Groups with the Inductive Limit Topologies[a]

OLGA V. SIPACHEVA

Department of General Topology and Geometry
Mechanics and Mathematics Faculty
Moscow State University
119899 Moscow, Russia

ABSTRACT: Countable spaces with an only nonisolated point for which the free topological groups are the inductive limits of the sets of words having restricted length are characterized. It is proved for a collectionwise normal X that if its free topological group is the inductive limit of the sets of words having restricted length, then any closed discrete set of non-P-points in X is countable. All the results obtained are valid for both Abelian and non-Abelian free groups.

The free (Abelian) topological group $F(X)$ (respectively, $A(X)$) of a completely regular T_1 space X in Markov's sense is the free (Abelian) algebraic group of X with the strongest group topology such that it induces the original topology of X. In other words, any continuous map of X to an arbitrary (Abelian) topological group G can be extended to a continuous homomorphism of $F(X)$ (respectively, of $A(X)$) to G.

Free topological groups and free Abelian topological groups were introduced and first investigated by Markov [3], [4]. Algebraically, the free group $F(X)$ of X is the set of words

$$g = g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n},$$

where n is a positive integer or zero (in the latter case the word g is empty), $\varepsilon_i = \pm 1$, and $g_i \in X$ for $i = 1, \dots, n$. So, every nonempty word is the product of letters, or the elements of the alphabet $X \cup X^{-1}$, where X^{-1} is a homeomorphic copy of X such that $X \cap X^{-1} = \varnothing$. The free Abelian group $A(X)$ looks similar: it consists of words

$$g = \varepsilon_1 g_1 + \dots + \varepsilon_n g_n,$$

whose letters are elements of the disjoint union of X and its homeomorphic copy $-X$.

Let $g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}$ be the reduced form of word g in $F(X)$. The number n is the length of g denoted $l(g)$. We use the designation $F_n(X)$ for the set of all words in $F(X)$ the length of which does not exceed n with the topology induced by $F(X)$. The similar meaning is assigned to $A_n(X)$: it is the set of all words in $A(X)$ whose

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irreducible forms comprise at most n letters with the topology inherited from $A(X)$.

A space $X = \bigcup_{n\in\omega} X_n$ has the *inductive limit topology* (or is the *inductive limit* of the sequence $\{X_n\}$), if a subset U of X is open (closed) in X whenever $U \cap X_n$ is open (closed, respectively) in X_n for all n. The inductive limit topology is also called direct limit topology or weak union topology.

The paper is concerned with a well-known and still unsolved problem in the free topological groups theory:

When is $F(X)$ ($A(X)$) the inductive limit of $\{F_n(X)\}$ (respectively, of $\{A_n(X)\}$)?

Apparently, the problem was first formulated explicitly by Pestov and Tkachenko in 1985 [11], but it was tackled as early as in 1948 by Graev [2], who proved that the free topological group of a compact space has the inductive limit topology. Then Mack, Morris, and Ordman [5] proved that the same is true for k_ω-spaces. Apparently, the strongest result in this direction was obtained by Tkachenko [10], who proved that if X is a P-space or if X is a so-called C_ω-space, that is, X is the inductive limit of an increasing sequence $\{X_n\}$ of its closed subsets such that all the finite powers of each X_n are countably compact and strictly collectionwise normal, then $F(X)$ has the inductive limit topology. In this paper, countable spaces with an only nonisolated point whose free topological groups have the inductive limit topologies are characterized. The same characterization is valid for free Abelian groups. It provides an example of a countable space with an only nonisolated point that does not contain infinite compact subspaces and whose free and free Abelian groups have the inductive limit topologies. Some other results concerning free groups with the inductive limit topologies are also presented.

We mostly will consider Abelian groups. All the necessary conditions obtained for free Abelian groups are also true for non-Abelian groups due to the following simple fact:

CLAIM 1: Let X be a completely regular T_1-space such that $F(X)$ is the inductive limit of $\{F_n(X)\}$. Then $A(X)$ is the inductive limit of $\{A_n(X)\}$.

Proof: Let $S \subseteq A(X)$ be such that for all n the set $S_n = S \cap A_n(X)$ is open in $A_n(X)$. Let us show that S is open. Consider the natural homomorphism $h\colon F(X) \to A(X)$. Put $U = h^{-1}(S)$ and $U_n = h^{-1}(S_n) \cap F_n(X)$ for every n. For each n fix an open in $A(X)$ set V_n such that $V_n \cap A_n(X) = S_n$. Because

$$U_n = h^{-1}(V_n \cap A_n(X)) \cap F_n(X)$$
$$= (h^{-1}(V_n) \cap h^{-1}(A_n(X))) \cap F_n(X) = h^{-1}(V_n) \cap F_n(X),$$

each set U_n is open in $F_n(X)$. On the other hand,

$$U_n = h^{-1}(S \cap A_n(X)) \cap F_n(X) = (h^{-1}(S) \cap h^{-1}(A_n(X))) \cap F_n(X) = U \cap F_n(X)$$

and $F(X)$ is the inductive limit of $\{F_n(X)\}$, therefore, U is open in $F(X)$. The homomorphism h is open, because the topology of $A(X)$ is the strongest one, hence, $S = h(U)$ is open. ❑

In what follows all spaces are supposed to be Tychonoff, all neighborhoods and covers open, and all pseudometrics continuous.

Elements and subsets of $A(X)$ are typed mostly in boldface to distinguish them from those of X. For example, $\mathbf{g} = \varepsilon_1 g_1 + \ldots + \varepsilon_n g_n$ represents word $\mathbf{g}$ as the sum of letters $\varepsilon_i g_i$, while $\mathbf{g} = \mathbf{u} + \mathbf{v}$ represents this word as the sum of words $\mathbf{u}$ and $\mathbf{v}$. The zero of $A(X)$, the empty word, is denoted as $\mathbf{0}$.

Letters k, l, m, n, p, q, r, s, and t stand for positive integers.

Let γ_1 and γ_2 be covers of a set X. The relation $\gamma_1 \prec \gamma_2$ means that γ_1 is a refinement of γ_2.

For a pseudometric d on X, a positive number ε, and $x \in X$

$$B_{d(x,\varepsilon)} = \{y \in X : d(x, y) < \varepsilon\}.$$

We need the following description of the topology of $A(X)$ (recall that all pseudometrics are supposed to be continuous):

CLAIM 2: (See [9].) Let X be a space. For a pseudometric d on X, put

$$U(d) = \{x_1 - y_1 + x_2 - y_2 + \ldots + x_n - y_n : n \in \omega, \ \sum_{i=1}^{n} d(x_i, y_i) < 1\}.$$

The family

$$\{U(d) : d \text{ is a pseudometric on } X\}$$

constitutes a base at zero of the topology of $A(X)$.

THEOREM 1: Let X be a space such that:
(a) $F(X)$ can be embedded into the Tychonoff product of metrizable groups as a subgroup;
(b) for any pseudometric d on X there exists a pseudometric $p \geq d$ on X such that for the metric space Y naturally associated with the pseudometric space (X, p) (that is, obtained from (X, p) by identifying points with zero p-distance), the free topological group $F(Y)$ is the inductive limit of $\{F_n(Y)\}$.

Then $F(X)$ is the inductive limit of $\{F_n(X)\}$.

Proof: Let $\mathbf{S} \subseteq F(X)$ be such that for all n the set $\mathbf{S}_n = \mathbf{S} \cap F_n(X)$ is open in $F_n(X)$. We have to prove that $\mathbf{S}$ is open. To this end, it suffices to show that any $\mathbf{g} \in \mathbf{S}$ belongs to the interior of $\mathbf{S}$. Take an arbitrary $\mathbf{g} \in \mathbf{S}$. The openness of all $\mathbf{S}_n$ in $F_n(X)$ implies that for each n there exists a neighborhood $\mathbf{U}_n$ of the identity such that $\mathbf{g} \cdot \mathbf{U}_n \cap F_n(X) \subseteq \mathbf{S}_n$ (if $\mathbf{g} \notin \mathbf{S}_k$, i.e., $l(\mathbf{g}) \geq k$, then $\mathbf{U}_k$ may be chosen so that $\mathbf{g} \cdot \mathbf{U}_k \cap F_k(X) = \varnothing$, because all $F_k(X)$ are closed in $F(X)$ [4]). Condition (a) implies that there exists a pseudometric d_n on X such that the interior of the set $\mathbf{U}_n$ in the topology of the free topological group of (X, d_n) contains the identity. There exists a pseudometric d on X, with respect to which all the pseudometrics d_n are continuous. By virtue of (b), we can assume that $F(X, d)$ has the inductive limit topology. Hence, the interior of $\mathbf{S}$ in $F(X, d)$ (the more so, in $F(X)$) contains $\mathbf{g}$. ❑

In particular, condition (a) is fulfilled for any space such that the cardinality of any discrete system of its open subsets is at most $\aleph_0$ [1].

The similar statement is true for Abelian groups; the proof is the same. Note that any Abelian group can be embedded into the product of metrizable groups as a subgroup, hence, for free Abelian groups the condition (a) should be omitted:

THEOREM 1′: Let X be a space and assume that for any pseudometric d on X there exist a pseudometric $p \geq d$ on X such that the free Abelian topological group $A(Y)$, where Y is the metric space naturally associated with (X, p), is the inductive limit of $\{A_n(Y)\}$. Then $A(X)$ is the inductive limit of $\{A_n(X)\}$.

Theorems 1 and 1′ have the following corollary (though, it can easily be derived from the fact that $F(X)$ has the inductive limit topology for any compact X [2] and that $F(X)$ is naturally embedded into $F(\beta X)$ for a pseudocompact X [6]):

COROLLARY: If a space X is pseudocompact, then $F(X)$ and $A(X)$ have the inductive limit topologies.

THEOREM 2: Let X be a countable space with only one nonisolated point $*$. The group $F(X)(A(X))$ is the inductive limit of $\{F_n(X)\}$ (respectively, of $\{A_n(X)\}$) if and only if for any collection $\{U_n : n \in \omega\}$ of open neighborhoods of $*$ there exists a neighborhood V of $*$ such that $V \cap (U_n \setminus U_{n+1})$ is finite for all n.

LEMMA 1: Let X be a countable space with only one nonisolated point $*$. Suppose there exists a collection $\{U_n : n \in \omega\}$ of open neighborhoods of $*$ such that $U_{n+1} \subseteq U_n$ for each n, and for any open neighborhood V of $*$ there is n such that $|V \cap (U_n \setminus U_{n+1})| = \aleph_0$. Then $A(X)$ is not the inductive limit of $\{A_n(X)\}$.

Proof: Without loss of generality, we can assume that $U_0 = X$ and the set $U_n \setminus U_{n+1}$ is infinite for each n. Enumerate

$$U_n \setminus U_{n+1} = \{x_{ni} : i \in \omega\}$$

and put

$$F_n = \{(x_{nm} - *) + n(x_{ij} - *) : i, j, m \in \omega,\ n < i < j < m\}$$

for all $n \in \omega$. We show that each F_n is a closed discrete subset of $A(X)$. Let $Y = \{*\} \cup \{x_{nm} : m \in \omega\}$ and $r : X \to Y$ be the retraction that maps $X \setminus Y$ to $\{*\}$. Clearly, Y is discrete and the map r is continuous. Let $\hat{r}: A(X) \to A(Y)$ be the canonical homomorphic extension of the map r; then $\hat{r}$ continuously maps $A(X)$ onto the discrete space $A(Y)$. For any $g \in A(Y)$ the set $\hat{r}^{-1}(g) \cap F_n$ is finite: if $\hat{r}^{-1}(g) \cap F_n$ is nonempty, then we have $g = \hat{r}((x_{nm_0} - *) + n(x_{i_0 j_0} - *))$ for some $m_0, i_0, j_0 \in \omega$ such that $n < i_0 < j_0 < m_0$, whence $g = x_{nm_0} - *$ and

$$\hat{r}^{-1}(g) \cap F_n = \{(x_{nm_0} - *) + n(x_{ij} - *) : i, j \in \omega,\ n < i < j < m_0\}.$$

Therefore, F_n is a closed discrete subspace of $A(X)$.

Since $F_n \subseteq A_{2n+2}(X) \setminus A_{2n+1}(X)$ for all n the set $F = \bigcup_n F_n$ is closed in the inductive limit topology. To prove the lemma, it is sufficient to show that $\mathbf{0} \in \overline{F}$, i.e., that for any pseudometric d on X we have $U(d) \cap F \neq \varnothing$.

Take an arbitrary (continuous) pseudometric d on X. As $B_d(*, ½)$ is a neighborhood of $*$, it follows from the conditions of the lemma that the set $M = \{m \in \omega:$

$d(*, x_{nm}) < ½)\}$ is infinite for some $n \in \omega$. As $B_d(*, 1/2n) \cap U_{n+1}$ is a neighborhood of $*$, it follows from the conditions of the lemma that the set $J = \{j \in \omega: d(*, x_{ij}) < 1/2n\}$ is infinite for some $i > n$. Take $j \in J$ such that $j > i$ and $m \in M$ such that $m > j$. Then $g = (x_{nm} - *) + n(x_{ij} - *) \in F_n$. We also have $g \in U(d)$, because

$$d(*, x_{nm}) + n \cdot d(*, x_{ij}) < \frac{1}{2} + n\frac{1}{2n} = 1.$$

Therefore, $g \in F_n \cap U(d)$. ❑

LEMMA 2: Let X be a countable space having an only nonisolated point $*$, and let for any collection $\{U_n: n \in \omega\}$ of open neighborhoods of $*$ there exist a neighborhood V of $*$ such that $V \cap (U_n \backslash U_{n+1})$ is finite for all n. Then $F(X)$ is the inductive limit of $\{F_n(X)\}$.

Proof: Note that the condition of the lemma can be reformulated in the following way: for any pseudometric d on X there exists a neighborhood V of $*$ such that the metric space $(V, d \restriction V)$ is compact. This implies that for any pseudometric d there exists a pseudometric $p \geq d$, for which (X, p) is locally compact. The space (X, p) is also σ-compact, because it is countable, therefore, it is a k_ω-space and $F(X, p)$ has the inductive limit topology [5]. Theorem 1 can be applied. ❑

Proof of Theorem 2. Combine the lemmas and Claim 1. ❑

To any filter p on ω there corresponds a space $\omega_p = \omega \cup \{p\}$: ω is its disrete subspace and neighborhoods of p are elements of the filter. Any countable space with one nonisolated point can be represented as ω_p for some filter p on ω. Note that Theorem 2 can be reformulated as follows:

THEOREM 2′: Let p be a filter on ω. The group $F(\omega_p)(A(\omega_p))$ is the inductive limit of $\{F_n(\omega_p)\}$ (respectively, of $\{A_n(\omega_p)\}$) if and only if for any collection $\{M_n: n \in \omega\} \subseteq p$ there exists $M \in p$ such that $M \backslash M_n$ is finite for all n.

Filters satisfying the condition of Theorem 2′ are called P-filters. An ultrafilter p on ω is a P-ultrafilter when and only when p is a P-point in $\beta\omega\backslash\omega$ (see [8]). Hence, we have:

COROLLARY: Let p be an ultrafilter on ω. The group $F(\omega_p)$ $(A(\omega_p))$ is the inductive limit of $\{F_n(\omega_p)\}$ (respectively, of $\{A_n(\omega_p)\}$) if and only if p is a P-point in $\beta\omega\backslash\omega$.

Professor W. Just kindly provided me with the following example:

EXAMPLE: There exists a P-filter p on ω such that all compact subspaces ω_p are finite.

Proof: Let

$$p = \left\{a \subseteq \omega: \sum_{n \in \omega\backslash a} \frac{1}{n+1} < \infty\right\}.$$

Consider an arbitrary infinite subset b of ω. There exists an infinite $c \subseteq b$, for which the series $\sum_{n \in c} 1/(n+1)$ converges. We have: $a = \omega \backslash c \in p$ and $b \backslash a$ is infinite, therefore, $b \cup \{p\}$ cannot be compact.

Show that p is a P-filter. Let $\{a_n : n \in \omega\}$ be a decreasing sequence of elements of p; then $\{b_n = \omega \backslash a_n : n \in \omega\}$ is an increasing sequence of subsets of ω such that $\sum_{n \in b_n} 1/(n+1) < \infty$. For each n choose $c_n \subseteq b_n$ such that $b_n \backslash c_n$ is finite and $\sum_{n \in c_n} 1/(n+1) \le 1/n^2$. Put $c = \bigcup_{n \in \omega} c_n$. Then $\sum_{n \in c} /(n+1) \le \sum_{n \in \omega} 1/n^2 < \infty$, hence, $a = \omega \backslash c \in p$. For each n, we have

$$a \backslash a_n = (\omega \backslash c) \backslash (\omega \backslash b_n) = b_n \backslash c \subseteq b_n \backslash c_n,$$

which is finite. ❑

REMARK: Note that for the filter p given in the example, the space $X = \omega_p$ has the following properties:

(a) $F(X)$ and $A(X)$ are the inductive limits of $\{F_n(X)\}$ and $\{A_n(X)\}$, respectively;

(b) X is a countable space with an only nonisolated point;

(c) all (pseudo) compact subspaces of X are finite, hence, X is not a k-space and not a C_ω-space.

THEOREM 3: Let X be a collectionwise normal space and $A(X)$ be the inductive limit of $\{A_n(X)\}$. Then any closed discrete subspace of X containing no P-points in X is at most countable.

LEMMA 3: Let X be a space, $Y = \{*_\alpha : \alpha \in \omega_1\}$ be a subspace of X, and for each $\alpha \in \omega_1$ there be a decreasing family $\{U_{\alpha,n} : n \in \omega\}$ of clopen neighborhoods of $*_\alpha$ such that

$$\operatorname{Int} \bigcap_{n \in \omega} U_{\alpha,n} \not\ni *_\alpha \quad \text{and} \quad U_{\alpha,0} \cap U_{\beta,0} = \varnothing \quad \text{for } \beta \ne \alpha$$

and

$$X = \bigcup_{\alpha \in \omega_1} U_{\alpha,0}.$$

Then $A(X)$ is not the inductive limit of $\{A_n(X)\}$. ❑

For each $\alpha \in \omega_1 \backslash \omega$, enumerate the set α as $\{\beta_{\alpha,i} : i \in \omega\}$. Put $C_{\alpha,n} = U_{\alpha,n} \backslash U_{\alpha,n+1}$ for $\alpha \in \omega_1$ and

$$F_{\alpha,n} = \{x - *_\alpha + n(y - *_{\beta_{\alpha,i}}) : x \in C_{\alpha,n},\ y \in C_{\beta_{\alpha,i},m}, \text{ and } n < m < i\}$$

$$F_n = \bigcup_{\alpha \in \omega_1 \backslash \omega} F_{\alpha,n}$$

for $\alpha \in \omega_1$ and $n \in \omega$. We show that each set F_n is closed in $A(X)$. Let $n \in \omega$. Put

$$\gamma = \{U_{\alpha,n} : \alpha \in \omega_1\} \cup \{C_{\alpha,k} : k < n,\ \alpha \in \omega_1\} \quad \text{and}$$

$$H = \{\textstyle\sum_{i<k}(x_i - y_i) : \text{ for all } i \text{ there is } O_i \in \gamma \text{ such that } x_i, y_i \in O_i,\ k \in \omega\}.$$

Since γ is a partitioning of the space X into clopen subsets, H is a clopen subgroup of $A(X)$. If $g \in F_{\alpha,n}$, then $F_{\alpha,n} \subseteq g + H$ and the sets $F_{\alpha,n} + H$ and $F_{\beta,n} + H$ are disjoint for distinct $\alpha, \beta \in \omega_1 \backslash \omega$. Therefore, each element of the clopen partitioning $\{g+H: g\in A(X)\}$ of the space $A(X)$ meets at most one set $F_{\alpha,n}$, and to prove the closedness of F_n in $A(X)$ it suffices to show that $F_{\alpha,n}$ is closed in $A(X)$ for each $\alpha\in\omega_1\backslash\omega$. Put

$$\theta_* = \{C_{\alpha,n}, U_{\alpha,0}\backslash C_{\alpha,n}\} \cup \{C_{\beta_{\alpha,i},m}, U_{\beta_{\alpha,i},0}\backslash C_{\beta_{\alpha,i},m}: n<m<i\},$$

$$\theta = \{X\backslash\bigcup\theta_*\}\cup\theta_*, \quad \text{and}$$

$$P = \{\textstyle\sum_{i<k}(x_i-y_i): \text{ for all } i \text{ there is } O_i\in\theta \text{ such that } x_i, y_i\in O_i, k\in\omega\}.$$

Since θ is a partitioning of the space X into clopen subsets, P is a clopen subgroup of $A(X)$. If $g\in A(X)$ and the set $M = F_{\alpha,n}\cap(g+P)$ is nonempty, then we have

$$M = \{x - *_\alpha + n(y - *_{\beta_{\alpha,i}}): x\in C_{\alpha,n}, \ y\in C_{\beta_{\alpha,i},m}\}$$

for some $m,i\in\omega$, $n<m<i$ and therefore, M is closed in $A(X)$. The family $\{g+P: g\in A(X)\}$ is a clopen partitioning of the space $A(X)$, hence $F_{\alpha,n}$ is closed in $A(X)$.

Clearly, $F_n\subseteq A_{2n+2}(X)\backslash A_{2n}(X)$, and as is shown above, F_n is closed in $A(X)$. Therefore, F_n is closed in $A_{2n+2}(X)$. Put

$$F = \bigcup_{n\in\omega} F_n.$$

To complete the lemma proof, it remains to show that $\mathbf{0}\in F$, *i.e.*, that for any (continuous) pseudometric d on X the neighborhood $U(d)$ of unity intersects the set F.

For each α there is n_α such that $B_d(*_\alpha, 1/2)$ intersects the set C_{α,n_α}. We may therefore fix n such that $A=\{\alpha: n=n_\alpha\}$ is uncountable. Using the sets $C_{\alpha,n}$ once more we can find an uncountable set $A'\subseteq A$ and $m>n$ such that $B_d(*_\alpha,1/2n)$ meets $C_{\alpha,m}$ for all $\alpha\in A'$.

Choose $\alpha\in A'$ such that $A'\cap\alpha$ is infinite, and take $i>m$ such that $\beta_{\alpha,i}\in A'$. Finally pick $x\in B_d(*_\alpha, 1/2)\cap C_{\alpha,n}$ and $y\in B_d(*_{\beta_{\alpha,i}}, 1/2n))\cap C_{\beta_{\alpha,i},n}$. It is readily verified that

$$x - *_\alpha + n(y - *_{\beta_{\alpha,i}})\in U(d)\cap F_n,$$

which completes the proof. ❑

LEMMA 4: Let x be a non-P-point of the regular space X. Then there are a closed subspace Y of X and a decreasing sequence $\{C_n\}_{n\in\omega}$ of clopen sets in Y such that $x\notin \mathrm{Int}_Y\bigcap_n C_n$.

Proof: Let $\{U_n\}_{n\in\omega}$ be a decreasing sequence of neighborhoods of x such that $\overline{U}_{n+1}\subseteq U_n$ for all n and $x\notin \mathrm{Int}\bigcap_n U_n$. For each n put $D_n=\overline{U}_n\backslash U_{n+1}$. Let $Y_1 = \{x\}\cup\bigcup_{n\text{ odd}} D_n$ and $Y_2=\{x\}\cup\bigcup_{n\text{ even}} D_n$.

Then Y_1 and Y_2 are closed and either

$$x\notin \mathrm{Int}_{Y_1}\bigcap_n(U_n\cap Y_1) \text{ or } x\notin \mathrm{Int}_{Y_2}\bigcap_n(U_n\cap Y_2).$$

For if O_1 and O_2 are open sets containing x such that $O_i\cap Y_i\subseteq\bigcap_n(U_n\cap Y_i)$ for $i=1,2$ then $O_1\cap O_2\subseteq\bigcap_n U_n$, a contradiction.

So one of Y_1 and Y_2 may be chosen as Y and we can then let $C_n=U_n\cap Y$ for all n. ❑

LEMMA 5: Let X be a collectionwise normal space and $Y = \{*_\alpha : \alpha \in \omega_1\}$ be a closed discrete subspace of X which does not contain P-points in X. Then there exists a closed subspace Z of X such that for each $\alpha \in \omega_1$ there is a decreasing family $\{U_{\alpha,n} : n \in \omega\}$ of clopen in Z neighborhoods of $*_\alpha$, for which

$$\operatorname{Int} \bigcap_{n\in\omega} U_{\alpha,n} \not\ni *_\alpha \quad \text{and} \quad U_{\alpha,0} \cap U_{\beta,0} = \varnothing \quad \text{for} \quad \beta \neq \alpha$$

and

$$Z = \bigcup_{\alpha\in\omega_1} U_{\alpha,0}.$$

To prove the lemma, it suffices to take a discrete family $\{U_\alpha \notin *_\alpha : \alpha \in \omega_1\}$ of open sets in X (it exists, because X is collectionwise normal), apply Lemma 4 to each of the sets U_α and replace it with a closed set containing $*_\alpha$ which has the desired decreasing sequence of clopen sets.

Proof of Theorem 3: Suppose, there is an uncountable closed discrete subspace of X consisting of non-P-points. According to Lemmas 3 and 5, there is a closed subspace Z of X such that $A(Z)$ is not the inductive limit of $\{A_n(Z)\}$. As X is collectionwise normal and Z is closed in X, Z is P-embedded in X, that is, any continuous pseudometric defined on Z can be extended to a continuous pseudometric on X [7]. It follows [9] that $A(Z)$ is naturally embedded in $A(X)$ as a closed subgroup. Let F be any subset of $A(Z)$ such that all the intersections $F \cap A_n(Z)$ are closed. Then F has the same property in $A(X)$: all the intersections $F \cap A_n(X)$ are closed. Therefore, F is closed in $A(X)$ and in $A(Z)$. Thus, $A(Z)$ is the inductive limit of $A_n(Z)$, which is not so. ❑

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The G_δ-topology, Light Compactness, and Pseudocompleteness

AARON R. TODD

Department of Mathematics
Baruch College, City University of New York
17 Lexington Avenue
New York, New York 10010

ABSTRACT: Baire category arguments generally support far stronger properties. In particular, these properties, such as Oxtoby's pseudocompleteness, are generally productive, whereas the Baire property is not. We show in the main theorem that the G_δ-expansion of a regular lightly compact space is pseudocomplete. This leads to intrinsic proofs of some cardinality results of Comfort and Robertson for pseudocompact topological groups—they had noted and used the fact that the G_δ-expansion of such a group topology is Baire. In particular, these new proofs avoid the Weyl completion, which, in this context, is the Stone-Čech compactification. The paper also contains an example of a topological group of smallest cardinality whose G_δ-expansion is not Baire, as well as some related questions.

As this work assumes no separation properties, we use the descriptive term G_δ-space for any topological space whose topology is closed under countable intersections. For $T_{3½}$-spaces, these are usually termed P-spaces (see [6]). For a topological space $(X, \mathcal{T})$, let $G_\delta\mathcal{T}$ be the weakest topology stronger than $\mathcal{T}$ for which $(X, G_\delta\mathcal{T})$ is a G_δ-space; in fact, $G_\delta\mathcal{T}$ is the expansion of the original topology by its G_δ-sets. We follow Bagley, Connell, and McNight, Jr. [2] in calling a topological space *lightly compact* if every locally finite collection of open sets is finite; this is equivalent to the property, each countable filterbase $\mathcal{U}$ of open sets has a cluster point, i.e., $\bigcap\{\mathrm{cl}\, U : U \in \mathcal{U}\} \neq \varnothing$ (see [7]). Lightly compact spaces are also called feebly compact (see [12]). A topological space is a *Baire* space if the intersection of each countable family of dense open sets is dense. The pseudocompleteness of J.C. Oxtoby [11] is convenient in that it includes all the standard properties that imply that a topological space is a Baire space. In particular, compact Hausdorff and completely metrizable spaces are pseudocomplete. As modified in [13], pseudocompleteness also has an illuminating version for bitopological spaces; this is discussed in [8]. We will give the needed definitions for pseudocompleteness shortly. By a *pseudocompact* space is meant a topological space on which every continuous real-valued function is bounded.

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In his handbook article [3], W.W. Comfort discusses and proves the following results from Comfort and Robertson [4] on density character $d(PG)$ and its cofinality $\mathrm{cf}(d(PG))$.

THEOREM: (Comfort and Robertson) Let G be a nondiscrete pseudocompact topological group. Then

(i) $|G| \geq c$;

(ii) $d(PG) \geq c$; and

(iii) $\mathrm{cf}(d(PG)) > \omega$.

The proofs make significant and interesting use of the Weyl completion $\overline{G}$, which, in this context, is the Stone-Čech compactification βG of G. This emphasis is quite understandable in an article on topological groups and well illustrates the "unusual luxury" of being able to identify and use this particular maximal compactification. Sections 6.12 and 6.13 in [3] contain discussions of these matters and outline some alternative results and proofs, all still of an extrinsic nature.

The first two conclusions of the above theorem follow directly from Corollary 5 below. In the original proof the second follows from an interesting and involved application of the first; in the proof here, the two follow from a lower bound on cellularity $c(PG)$. Originally only the proof of the third conclusion used the fact that PG is a Baire space; here the proof of the first two involve pseudocompleteness of PG, which implies the Baire result. This property is established in Corollary 3 below. All proofs of theorems here are intrinsic, quite elementary in technique, apply to rather general spaces, and involve the pseudocompleteness property.

The following lemma is crucial for proof of the main theorem. To prove it, we use the open filterbase characterization of light compactness. By a *strongly nested* sequence of sets we mean a sequence $(C_n)_n$ for which each $\mathrm{int}C_n$ contains $\mathrm{cl}C_{n+1}$. Also note, for later use, that a regular closed set inherits the lightly compact property (see [2, Theorem 14]).

LEMMA 1: If a regular open set $U = \mathrm{int\,cl}U$ of a lightly compact space $(X, \mathcal{T})$ contains the intersection $F = \bigcap_n C_n$ of a strongly nested sequence $(C_n)_n$, then U contains all but a finite number of terms of the sequence $(C_n)_n$.

Proof: If, for some n, $\mathrm{int}\,C_n \backslash \mathrm{cl}U = \varnothing$, then $C_{n+1} \subset U$, and the conclusion holds. Suppose there is no such n, then $(\mathrm{int}\,C_n \backslash \mathrm{cl}U)_n$ is a countable filterbase of open sets, and, as $(X, \mathcal{T})$ is lightly compact, $\bigcap_n \mathrm{cl}(\mathrm{int}\,C_n \backslash \mathrm{cl}U) \neq \varnothing$. Now, for each n, $C_n \backslash U \supset \mathrm{cl}C_{n+1} \backslash U$, so that $C_n \backslash U \supset \mathrm{cl}(\mathrm{int}\,C_{n+1} \backslash \mathrm{cl}U)$. Therefore, $\varnothing = F \backslash U \supset \bigcap_n \mathrm{cl}(\mathrm{int}\,C_{n+1} \backslash \mathrm{cl}U) \neq \varnothing$, a contradiction. ❑

By a *pseudobase* for a topological space $(X, \mathcal{T})$ we mean a collection $\mathcal{B}$ of subsets such that (i) for each $B \in \mathcal{B}$, $\mathrm{int}\,B \neq \varnothing$ and (ii) for each nonempty $U \in \mathcal{T}$ there is a $B \in \mathcal{B}$, with $U \supset B$. (If all elements of $\mathcal{B}$ are open, it would be a pseudobase according to [11] or a π-base according to other authors.) We say a topological space is *quasiregular* if each nonempty open set contains the closure of a nonempty open set; equivalently, there is a pseudobase of closed sets. A quasiregular space $(X, \mathcal{T})$ is *pseudocomplete* if there is a sequence $(\mathcal{B}_n)_n$ of pseudobases such that $\bigcap_n B_n \neq \varnothing$, whenever each $B_n \in \mathcal{B}_n$ and $\mathrm{int}\,B_n \supset \mathrm{cl}B_{n+1}$, for $n = 1, 2, \ldots$. (Henriksen *et al.* [8] show that, with the above definition of pseudobase, the strong nesting of $(B_n)_n$ may be replaced by nesting, i.e., each $B_n \supset B_{n+1}$.) It is easily verified that a pseudocomplete space is a Baire space (see [11], [13]).

The main theorem applies these properties to the G_δ-expansion of a topology.

MAIN THEOREM 2: If each point of a regular topological space $(X,\mathcal{T})$ has a lightly compact neighborhood, then the G_δ-space $(X,G_\delta\mathcal{T})$ has a base of clopen sets such that each descending sequence of nonempty members of this base has a nonempty intersection. Consequently, $(X, G_\delta\mathcal{T})$ is a pseudocomplete space, and so also a Baire space.

Proof: For $x \in U \in \mathcal{T}$, let C be a lightly compact neighborhood of x. By regularity, there is an open neighborhood V of x with $\mathrm{cl}V \subset U \cap \mathrm{int}\, C$. As $(C, \mathcal{T}|C)$ is lightly compact, $(\mathrm{cl}V, \mathcal{T}|\mathrm{cl}V)$ is also by the remark before Lemma 1. Therefore, each point has a neighborhood base of regular closed lightly compact sets. Therefore, the collection $\mathcal{C}$ of nonempty regular closed lightly compact sets is a pseudobase of closed sets and provides a base $\{\mathrm{int}\, C : C \in \mathcal{C}\} \cup \{\varnothing\} \subset \mathcal{T}$ for the original topology. Let $\mathcal{B}$ consist of all $B = \bigcap_n C_n$ where each $C_n \in \mathcal{C}$ and $(C_n)_n$ is strongly nested. Note that $\mathcal{B}$ is closed under finite nonempty intersections. Since the sequence $(C_n)_n$ of closed sets is strongly nested, the set $B = \bigcap_n C_n = \bigcap_n \mathrm{int}\, C_n$ is a closed G_δ-set which is nonempty because C_1 is lightly compact. Thus $\mathcal{B}$ is a collection of nonempty clopen sets of $(X, G_\delta\mathcal{T})$.

Suppose $x \in V \in G_\delta\mathcal{T}$. There is a sequence $(U_n)_n$ in $\mathcal{T}$ with $x \in \bigcap_n U_n \subset V$. By an induction argument, there is a strongly nested sequence $(C_n)_n$ of closed neighborhoods of x in $\mathcal{C}$ with $\bigcap_{k<n+1} U_k \supset U_{n+1} \cap \mathrm{int}\, C_n \supset C_{n+1}$. Therefore, $B = \bigcap_n C_n$ is an element of $\mathcal{B} \subset G_\delta\mathcal{T}$, and $x \in B = \bigcap_n C_n \subset \bigcap_n U_n \subset V$. Thus the collection $\mathcal{B} \cup \{\varnothing\}$ is a base of clopen sets for $(X, G_\delta\mathcal{T})$.

We need to show that each descending sequence $(B_n)_n$ in $\mathcal{B}$ has a nonempty intersection. For each n, there is a strongly nested sequence $(C_{n,m})_m$ in $\mathcal{C}$ with $B_n = \bigcap_m C_{n,m}$. An induction argument gives an increasing sequence $(m_n)_n$ such that, for each n,

$$C_{n+1, m_{n+1}} \subset (\mathrm{int}\, C_{n,m_n}) \cap (\textstyle\bigcap_{l<n+1} \mathrm{int}\, C_{l,n}).$$

The argument uses Lemma 1 to show the existence of the set on the left, since the set on the right contains B_{n+1} and is a regular open set. (To start the argument, we may define $m_1 = 1$.) Now $(C_{n,m_n})_n$ is a strongly nested sequence in $\mathcal{C}$ with each C_{n,m_n} containing B_n. Moreover,

$$B = \textstyle\bigcap_n B_n = \bigcap_n (\bigcap_{l<n+1} C_{l,n}) \supset \bigcap_n C_{n,m_n} \supset B,$$

thus $B = \bigcap_n C_{n,m_n}$ is in $\mathcal{B}$, and so is nonempty, as required.

Since $(X, G_\delta\mathcal{T})$ has a base $\mathcal{B} \cup \{\varnothing\}$ of clopen sets it is regular. Letting each $\mathcal{B}_n$ equal $\mathcal{B}$, we see that $(X, G_\delta\mathcal{T})$ is pseudocomplete. ❑

We may note that the proof actually establishes that $(X, B_\delta\mathcal{T})$ enjoys the countably subcompact property, a completeness type property stronger than pseudocompleteness (see [1]).

Theorem 3 of [2] states that a $T_{3½}$-space is lightly compact if and only if it is pseudocompact. So a topological group G is lightly compact if and only if it is pseudocompact. Following Comfort [3], we denote G with its G_δ-topology by PG.

In [3], PG is stated to be Baire if G is pseudocompact; the Weyl completion, $\overline{G}$, which, in this context is the Stone-Čech compatification βG, is suggested as the site of the proof. The above gives somewhat more and with no appeal to the Weyl completion $\overline{G}$, as follows.

COROLLARY 3: If G is a pseudocompact topological group, then PG is pseudocomplete and so is also a Baire space.

Proof: Direct. ❑

W.W. Comfort [3, 6.13] notes a result of E.K. van Douwen [5] that a nonempty pseudocompact $T_{3½}$-space X with no isolated points has cardinality at least c. The suggested proof uses a Cantor tree argument applied to βX. As the following shows, a Cantor tree argument may be applied directly to the space itself to obtain cardinality results not restricted to $T_{3½}$-spaces. Of course, the $T_{3½}$-space cardinality result of van Douwen then follows immediately, once it is realized that a quasiregular lightly compact space is pseudocomplete (see [13, Proposition 2.4]).

By a *rare* set we mean one whose closure has empty interior; this is also known as a nowhere dense set [9, p. 145]. We extend this to say that a *rare point* is a point whose singleton set is rare. (Note that a T_1-space with no isolated points has all singletons rare.)

THEOREM 4: A nonempty pseudocomplete space $(X, \mathcal{T})$ with a dense set of rare points satisfies $|X| \geq d(G_\delta\mathcal{T}) \geq c(G_\delta\mathcal{T}) \geq c$.

Proof: Only the last inequality involving the cellularity of the G_δ-expansion need be shown as the others always hold. We use the following: Suppose U is an nonempty open set. There is a rare point x in U. Since $\{x\}$ is rare, $U\backslash \mathrm{cl}\{x\}$ is a nonempty open set. As X is quasiregular, there is a nonempty open set U_0 with $\mathrm{cl}U_0 \subset U\backslash\mathrm{cl}\{x\}$. As $U\backslash\mathrm{cl}U_0$ contains x, it is a nonempty open set. Again by quasiregularity, there is a nonempty open set U_1 with $\mathrm{cl}U_1 \subset U \setminus \mathrm{cl}U_0$. Thus U_0, U_1 are nonempty open sets such that $\mathrm{cl}U_0$, $\mathrm{cl}U_1$ are disjoint subsets of U.

Suppose $(\mathcal{B}_n)_n$ is a sequence of pseudobases as required by the definition of pseudocompleteness. Let $B(\varnothing) = X$, $\mathcal{B}_0 = \{X\}$ and $n \geq 0$. Assume that the subsets $B(f)$ with $f \in \bigcup\{ {}^k2 : 0 \leq k < n\}$ satisfy: For each $0 \leq k < n$ and $f \in {}^k2$,

(i) $B(f \cup \{(k, 0)\})$ and $B(f \cup \{(k, 1)\})$ are members of $\mathcal{B}_{k+1}$, and

(ii) $\mathrm{cl}B(f \cup \{(k, 0)\})$ and $\mathrm{cl}B(f \cup \{(k, 1)\})$ are disjoint subsets of $\mathrm{int}\,B(f)$.

This may be extended with $n + 1$ replacing n as follows. Let $f \in {}^n2$. As $B(f) \in \mathcal{B}_n$, $\mathrm{int}\,B(f) \neq \varnothing$. From an earlier observation, there are nonempty open sets U_0, U_1 with $\mathrm{cl}U_0$, $\mathrm{cl}U_1$ disjoint subsets of $\mathrm{int}\,B(f)$. As $\mathcal{B}_{n+1}$ is a pseudobase, it has elements $B(f \cup \{(n, 0)\})$, $B(f \cup \{(n, 1)\})$ contained in U_0, U_1, respectively, whose closures are, necessarily, disjoint subsets of $B(f)$. As these exist for each $f \in {}^n2$, the induction step is complete, and we conclude that, for each $k \in \omega$ and $f \in {}^k2$, we have $B(f)$ satisfying (i) and (ii) above. Let $g \in {}^\omega2$. For each $n = 1, 2, \ldots, B(g|n) \in \mathcal{B}_n$ and $\mathrm{int}\,B(g|n) \supset \mathrm{cl}\,B(g|(n+1))$. Therefore, $\bigcap_n B(g|n) \neq \varnothing$. Let $B(g)$ be this nonempty G_δ-set. For distinct $g, h \in {}^\omega2$, $g|n$, $h|n$ are distinct for large enough n, so there is $n \in \omega$ with $B(g|n)$, $B(h|n)$ disjoint. Therefore $B(g)$ and $B(h)$ are disjoint. Thus, the collection $\{B(g) : g \in {}^\omega2\}$ has cardinality $|{}^\omega2| = c$, consists of disjoint nonempty open subsets of $(X, G_\delta\mathcal{T})$, and so its cellularity $c(G_\delta\mathcal{T})$ is greater than or equal to c. ❑

The following corollary of this theorem gives the first two conclusions of the theorem of Comfort and Robertson above.

COROLLARY 5: A nondiscrete Hausdorff topological group G that is locally pseudocompact satisfies $|G| \geq d(PG) \geq c(PG) \geq \mathfrak{c}$.

Proof: Let C be a pseudocompact, therefore lightly compact, neighborhood of the identity e. As G is regular, there is an open neighborhood U of e with $\mathrm{cl}U \subset \mathrm{int}C$. The nonempty $T_{3½}$-space $\mathrm{cl}U$ is lightly compact in its subspace topology. As it has no isolated points, points of $X = U$ are rare. Since X is a nonempty open subset of G, the corollary follows from Theorem 4, once we see that X is pseudocomplete. The sequence $(\mathcal{B}_n)_n$ with each $\mathcal{B}_n = \{\mathrm{cl}V\colon V$ is open and nonempty and $\mathrm{cl}V \subset U\}$ provides the required pseudobases as each member $\mathrm{cl}V$ is lightly compact. ❑

The following example embodies a quite general construction of topological spaces with a G_δ-expansion that is not Baire. The specific application provides an Abelian topological group G whose G_δ-expansion, PG, is an Abelian topological group that is not Baire. It also illustrates that a nondiscrete P-space may have cardinality $\omega_1 < \mathfrak{c}$ for models of set theory for which the continuum hypothesis is false. Of course a nonzero countable cardinality is not possible as each singleton would be a G_δ-set and so isolated in PX. This example arises from one of R.D. Kopperman [10] which the author greatly appreciates. The author is also happy to thank Paul Meyer for his informative discussions on this and many other topics, as well as the referee who encouraged a clarifying brevity throughout this paper.

EXAMPLE 6: Let G be the topological subgroup of $\mathbf{Z}_2{}^{\omega_1}$ consisting of all finitely nonzero elements, i.e., $G = \{x \in \mathbf{Z}_2{}^{\omega_1}\colon \{\alpha \in \omega_1\colon x(\alpha) \neq 0\}$ is finite$\}$ with its topology inherited from the product topology on $\mathbf{Z}_2{}^{\omega_1}$ where each factor, $\mathbf{Z}_2 = \{0,1\}$, has its discrete topology. The topology on G is generated by the neighborhood base at 0, $\mathcal{U} = \{U_F\colon F$ is a finite subset of $\omega_1\}$ where $U_F = \{x \in G : x(\alpha) = 0$ for $\alpha \in F\}$. As countable intersections of these contain terms of the descending ω_1-sequence, $U_\alpha = \{x \in G : x(\beta) = 0$ for $\beta < \alpha\}$, and these terms are countable intersections of members of $\mathcal{U}$, the collection $\mathcal{V} = \{U_\alpha\colon \alpha < \omega_1\}$ and its translates provide neighborhood bases at points of G for its G_δ-topology. This provides the topology for the P-space PG. Let $G_n = \{x \in G : |\{\alpha \in \omega_1 : x(\alpha) \neq 0\}| \leq n\}$; we shall say that the members of $G_n \backslash G_{n-1}$ are of *rank n*. As each member of G is finitely nonzero, $G = \bigcup_n G_n$. If x is of rank greater than n, then $x(\alpha) \neq 0$ for more than n elements $\alpha \in \omega_1$. Let $F = \{\alpha \in \omega_1\colon x(\alpha) \neq 0\}$ and let α_0 be an element of ω_1 larger than all elements of F. Now consider $x + U_{\alpha_0}$. Any member y of this PG-neighborhood of x must agree with x on all $\alpha < \alpha_0$; in particular, $y(\alpha) = x(\alpha) \neq 0$ for $\alpha \in F$. Since $|F| > n$, y is of rank greater than n, i.e., $y \in PG \backslash G_n$. Therefore, G_n is closed in PG. We now show its complement is PG-dense. Suppose x is in G_n, and define $x_\alpha \in G$, by requiring $x_\alpha(\beta)$ to equal 1 for $\alpha = \beta$ and, 0 otherwise. Also note, by an argument similar to the above, that for large enough $\alpha \in \omega_1$, $x + \sum_{k=1}^{n+1} x_{\alpha+k}$ is in the complement of G_n. Moreover, $\lim_{\alpha \to \omega_1}(x + \sum_{k=1}^{n+1} x_{\alpha+k})$ is x. Therefore, $G \backslash G_n$ is an open dense set of PG, yet $\bigcap_n (G \backslash G_n) = G \backslash \bigcup_n G_n = \varnothing$, so PG is a P-space that is not Baire. ❑

QUESTIONS 7: It is easy enough to find a pseudocomplete space with a stronger topology for which it is not a Baire space. (Any infinite dimensional Banach space together with its strongest locally convex topology furnishes an example.) What about one with a non-Baire G_δ-expansion? (A first countable space won't do as its G_δ-expansion is discrete.) In the same vein, a non-Baire space may have a stronger pseudocomplete topology. (An infinite dimensional Banach space with its weak topology satisfies this.) What about a non-Baire space with a pseudocomplete G_δ-expansion with no isolated points? The next and final question is more vague: As with the first two conclusions of the Comfort and Robertson theorem stated at the beginning, can the third be restated more generally (without algebraic hypotheses) and proved intrinsically using elementary techniques?

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On the Product of a Compact Space with an Absolutely Countably Compact Space

JERRY E. VAUGHAN

Department of Mathematics
The University of North Carolina at Greensboro
Greensboro, North Carolina 27412

ABSTRACT: We show that the product of a compact sequential T_2-space, with an absolutely countably compact T_3-space, is absolutely countably compact, and give several related results. For example, we show that every countably compact GO-space is absolutely countably compact, and that the product of a compact T_2-space of countable tightness with an absolutely countably compact, ω-bounded T_3-space (in particular a countably compact GO-space) is absolutely countably compact.

1. INTRODUCTION

A space is called *countably compact* provided every countable open cover has a finite subcover. A characterization of countable compactness (see [5, 3.12.22(d)] or [4]) states that a T_2-space X is countably compact iff for every open cover $\mathcal{U}$ of X there exists a finite set $F \subset X$ such that

$$St(F, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap F \neq \varnothing\} = X.$$

M.V. Matveev [11] defined a space X to be *absolutely countably compact* (acc) provided for every open cover $\mathcal{U}$ of X and every dense $D \subset X$, there exists a finite set $F \subset D$ such that $\mathrm{St}(F, \mathcal{U}) = X$, and noted:

$$\text{compact} \Rightarrow \text{acc} \overset{T_2}{\Rightarrow} \text{countably compact}.$$

It is an interesting fact that the acc property is not necessarily preserved by products with compact spaces (see [11, Example 2.2]). Concerning this, Matveev proved [11, Theorem 2.3] that if Y is a compact and first countable space, and X is an acc T_2-space, then $X \times Y$ is acc, and he raised the question:

QUESTION 1.1: (Matveev [11, Question 2.4].) Is $X \times Y$ acc provided Y is a compact space with countable tightness and X is an acc T_2-space?

We recall a few definitions. A space X has *countable tightness* provided that whenever $x \in \overline{A}$ there exists a countable $C \subset A$ such that $x \in \overline{C}$, and X is called a *sequential space* provided every sequentially closed set is closed (a set $A \subset X$ is *sequentially closed* if and only if A contains all limits of all convergent sequences

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from A). Every sequential space has countable tightness, see [5, 1.7.13(c)]. In Section 2, we prove our main result (in ZFC):

THEOREM 1.2: If Y is a compact sequential T_2-space, and X is an acc T_3-space, then $X \times Y$ is acc.

With Theorem 1.2 and the following well-known theorem, we can give an affirmative answer to Matveev's question provided we assume the proper forcing axiom [PFA] (Question 1.1 remains open in ZFC).

THEOREM 1.3: (Z. Balogh [2]) (PFA) Every compact Hausdorff space of countable tightness is sequential.

We also consider further conditions under which the product of a compact space with an acc space is acc. There may be some relevance of these conditions to Matveev's Question. If in some model of set theory there is a counterexample $X \times Y$ to Question 1.1, then Y must be a compact nonsequential space with countable tightness, and X must be an acc space which is not ω-bounded and does not have countable density-tightness (defined below). A space is called ω-*bounded* if every countable subset is contained in a compact set [9]. In Section 3 we prove:

THEOREM 1.4: $X \times Y$ is acc provided Y is a compact T_2-space of countable tightness and X is an acc, ω-bounded T_3-space.

We also obtain the next result concerning GO-spaces (generalized ordered spaces, i.e., spaces which are subspaces of some linearly ordered topological space; see [8]).

COROLLARY 1.5: $X \times Y$ is acc provided Y is a compact T_2-space of countable tightness and X is a countably compact GO-space.

Corollary 1.5 follows from Theorem 1.4, the following easy result Lemma 1.6, and the facts that every countably compact GO-space is ω-bounded [9, Theorem 3], and every GO-space is orthocompact [8, 5.23]. Recall that a space is called *orthocompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ such that for every $\mathcal{V}' \subset \mathcal{V}$, we have $\cap \mathcal{V}'$ is open [14], [8].

LEMMA 1.6: Every countably compact, orthocompact space is acc.

The converse of Lemma 1.6 is false. The Σ-product

$$X = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x(\alpha) = 1\}| \leq \omega\}$$

is acc [11, Theorem 1.9], but not orthocompact [14, Example 4.3].

These results also reveal an interesting class of acc spaces:

COROLLARY 1.7: Every countably compact GO-space is acc.

We mention that ω-bounded spaces need not be acc [11, Example 2.2].

DEFINITION 1.8: The *density-tightness* of a space X, denoted $d_t(X)$, is the smallest infinite cardinal κ such that for every dense subset $D \subset X$ and every $x \in X$ there exists $E \in [D]^\kappa$ such that $x \in \mathrm{cl}_X(D)$.

Clearly, $d_t(X) \leq t(X)$, and these need not be equal. For example, any space with a countable dense set of isolated points has countable density-tightness, but of course, need not have countable tightness. Countable density-tightness was introduced by Matveev using different terminology [11].

THEOREM 1.9: $X \times Y$ is acc provided Y is a compact T_2-space with countable density-tightness, and X is an acc T_3-space with countable density-tightness.

This theorem follows at once from the following two results.

LEMMA 1.10: (Matveev [11, Lemma 1.7]) If X is countably compact and has countable density-tightness then X is acc.

THEOREM 1.11: If Y is locally compact, then

$$d_t(X \times Y) \leq \max\{d_t(X), d_t(Y)\}.$$

The proof of Theorem 1.11 is similar to that of Malyhin's analogous result concerning tightness [10]. Since locally compact spaces need not be countably compact, one cannot replace "Y is compact" by "Y is locally compact" in Theorem 1.9. One can, however, replace "Y is compact" by "Y is locally compact and Y has property $\mathcal{P}$" where $\mathcal{P}$ is any property such that the product of a countably compact space with a space having property $\mathcal{P}$ is countably compact (for example, take $\mathcal{P}$ to be "sequential compactness," see [15, 3.4, 3.9]).

We conclude this section with several questions.

QUESTION 1.12: (A. Arhangel'skiĭ; see [11, 1.12].) Does countably compact + normal $\Rightarrow$ acc?

In light of Corollary 1.7, it now seems reasonable to ask the following two questions:

QUESTION 1.13: Does countably compact + monotonically normal $\Rightarrow$ acc?

QUESTION 1.14: (A. Bella) Does countably compact + radial $\Rightarrow$ acc?

NOTE ADDED IN PROOF: Both of these questions have been answered in the affirmative. See "From Countable Compactness to Absolute Countable Compactness," by Mary Ellen Rudin, Ian S. Stares, and Jerry E. Vaughan (to appear).

The results in this paper were announced in two preliminary reports [17], [18]. Further results about acc spaces can be found in [3], [12], [13], and [16].

2. PROOF OF THEOREM 1.2

We use the following standard notation: for a set D, $[D]^{<\omega}$ denotes the set of all finite subsets of D, and $[D]^\omega$ denotes the set of all finite or countable subsets of D. If D is a subset of a topological space X, the $\aleph_0$-*closure of D* [1] is

$$[D]_{\aleph_0} = \{x \in X: (\exists C \in [D]^{\omega})(x \in \overline{C})\}.$$

Proof (by contradiction): There exist an open cover $\mathcal{U}$ of $X \times Y$ and a dense set $D \subset (x, y)$ such that for all $G \in [D]^{<\omega}$, $St(G,\mathcal{U})$ does not cover (x, y). Since $X \times Y$ is countably compact (X is T_2), we may assume further that for all $G \in [D]^{\omega}$, $St(G,\mathcal{U})$ does not cover (x, y). Now we show that this inability to cover $X \times Y$ can be demonstrated by a single point of Y. For all $G \in [D]^{\omega}$ define $F_G = \{y \in Y: (\exists x \in X)\ ((x, y) \notin St(G, \mathcal{U})\}$. Since Y is a sequential space, and X is countably compact, π_Y is a closed mapping by a theorem of Fleischer and Franklin [6]. Thus each set F_G is closed in Y, since $F_G = \pi_Y((X \times Y) - St(G,\mathcal{U}))$ is a closed set in $(X \times Y)$. Thus

$$\mathcal{F} = \{F_G: G \in [D]^{\omega}\} \tag{1}$$

is a filter base of closed sets in Y; so by compactness there exists $y \in \bigcap \mathcal{F}$. Clearly,

$$\forall G \in [D]^{\omega}, St(G, \mathcal{U}) \text{ does not cover } X \times \{y\}. \tag{2}$$

CLAIM: For this y there exists a nonempty open set $V \subset X$ such that

$$(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} = \varnothing.$$

First we note that the $\aleph_0$-closure of D is not dense in $X \times \{y\}$: Otherwise if $E = (X \times \{y\}) \cap [D]_{\aleph_0})$ is dense in $X \times \{y\}$, then by acc, there exists a finite $E' \subset E$ such that $St(E', \mathcal{U})$ covers $X \times \{y\}$. For each $e \in E$ there is a countable $D_e \in [D]^{\omega}$ such that $e \in \mathrm{cl}(D_e)$, but this implies that

$$X \times \{y\} \subset St(E', \mathcal{U}) \subset St(\cup\{D_e: e \in E'\},\mathcal{U})$$

which contradicts (2). Hence, there exists a nonempty open $V \subset X$ such that

$$(V \times \{y\}) \cap [D]_{\aleph_0} = \varnothing.$$

Since X is regular, we may assume that

$$(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} = \varnothing. \tag{3}$$

This proves the Claim. Now we show that the set $Z = \pi_Y((\overline{V} \times Y) \cap [D]_{\aleph_0})$ is sequentially closed in Y: Let (z_i) be a sequence in Z that converges to a point $z \in Y$. For each $i \in \omega$ there exists $e_i \in (\overline{V} \times Y) \cap [D]_{\aleph_0}$ such that $\pi_Y(e_i) = z_i$. By countable compactness, there exists a limit point p of the sequence (e_i), and since (z_i) converges to z and Y is Hausdorff, we have $\pi_Y(p) = z$.

Clearly, $p \in (\overline{V} \times Y) \cap [D]_{\aleph_0}$, and therefore $z \in Z$. Thus, Z is sequentially closed, and since Y is sequential, Z is closed in Y. Since D is dense in $(X \times Y)$, hence in $V \times Y$, we see that Z is dense in Y; so $Z = Y$. Thus $y \in Z$, but this contradicts (3), and completes the proof. ❑

3. PROOF OF THEOREM 1.4

The proof requires only minor changes from that of Theorem 1.2 . To begin, we proceed as in the proof of Theorem 1.2 and construct the filter base $\mathcal{F}$ in (1), but this time we do not know that the sets F_G in $\mathcal{F}$ are closed in Y; so we cannot be sure that there exists $y \in \bigcap\mathcal{F}$. Instead, by compactness there exists

$$y \in \bigcap \{\overline{F_G} : G \in [D]^{\omega}\}.$$

CLAIM: For this y there exists a nonempty open set $V \subset X$ such that

$$(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} = \varnothing. \tag{4}$$

As before, we show that $[D]_{\aleph_0}$ is not dense in $X \times \{y\}$. Otherwise, by acc, there exists a finite $F \subset [D]_{\aleph_0}$ such that $St(F,\mathcal{U}) \supset X \times \{y\}$. Since each $x \in F$ is in $[D]_{\aleph_0}$ it follows that there exists $G \in [D]^{\omega}$ such that $St(G,\mathcal{U}) \supset X \times \{y\}$. Since $y \in \overline{F}_G$, by countable tightness of Y, there is a countable $Z \subset F_G$ such that $y \in \overline{Z}$. Let $Z = \{z_i : i < \omega\}$, and let $u \in \beta\omega\backslash\omega$ such that $y = u - \lim z_i$ (e.g., see [15, Section 4]). Pick $x_i \in X$ such that $(x_i, z_i) \notin \mathrm{St}(G,\mathcal{U})$. Since X is ω-bounded, there exists $x \in X$ such that $x = u - \lim x_i$, and thus $(x, y) = u - \lim(x_i, y_i)$. But this is impossible because $(x, y) \in St(G,\mathcal{U})$ and the $(x_i, y_i) \notin \mathrm{St}(G, \mathcal{U})$. By T_3, this proves the Claim.

Now let V satisfy (4). By countable tightness of Y there exists a countable $C = \{z_i : i < \omega\}$ such that

$$C \subset \pi_Y((V \times Y) \cap D) \text{ and } y \in \overline{C}.$$

For each $i \in \omega$ pick $x_i \in V$ so that $(x_i, z_i) \in ((V \times Y) \cap D)$. As in the proof of the Claim there exists $x \in \overline{V}$ and $u \in \beta\omega\backslash\omega$ such that $(x, y) = u - \lim(x_i, z_i)$, but this contradicts (4).

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Quotients of Complete Boolean Algebras

J. VERMEER
Faculty of Technical Mathematics and Informatics
TU Delft
Postbus 5031
2600 GA Delft, the Netherlands

ABSTRACT: We show that every κ-field of sets with $\kappa > 2^{\omega_0}$ is a quotient of a complete Boolean algebra.

INTRODUCTION

It was asked of Choquet [4] if every (weakly) countably complete Boolean algebra $\mathcal{B}$ is a quotient of a complete Boolean algebra. In [2] it was proved that the σ-complete Boolean algebra of the Baire sets of the unit interval is consistently not a quotientof a complete Boolean algebra. It is remarkable that not many "absolute" examples of (nontrivial) σ-complete Boolean algebra's are known to be a quotient of a complete Boolean algebra (for other reasons than the definition only). (What about the Lebesque sets of $[0,1]$?) The only results known to me in this area can be found in [1] and [3].

The goal of this note is to show that it is possible to enlarge the algebra of the Baire sets, in such a way that the algebra is a quotient of a complete Boolean algebra. Using the algebra, we obtain the following result.

THEOREM 0.1: Every $(2^{\omega_0})^+$- field of sets is a quotient of a complete Boolean algebra.

1. DEFINITIONS AND CONSTRUCTIONS

For all undefined notions, we refer to [3]. All spaces under consideration are assumed to be zero dimensional. The Boolean algebra of clopen sets of X is to be denoted by $CO(X)$. The smallest ω_1-field of sets containing $CO(X)$ is the algebra $Bair(X)$, the Baire sets of X.

Let κ be a cardinal number. A Boolean algebra $\mathcal{B}$ is called a κ-*field of sets* if the algebra has a representation as a subalgebra of a power set $\mathcal{P}(D)$ with the property:

$$A_i \in \mathcal{B}(i < \gamma < \kappa) \Rightarrow \bigcup\{A_i : i < \gamma\} \in \mathcal{B}.$$

A space X is called a P_κ-*space* if the intersection of less than κ open sets is again open. In this case the algebra $CO(X)$ is a κ-field of sets. P_{ω_1} spaces are called P-

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spaces. $\mathcal{R}o(2^\kappa)$ denotes the family of regular open sets of the space 2^κ and the Stone space of this algebra, the absolute of 2^κ, is denoted by $E(2^\kappa)$. The map Π: $E(2^\kappa) \to 2^\kappa$ is the natural (irreducible) projection defined by $\Pi(\mathcal{U}) = \cap\{\mathrm{cl}(U): U \in \mathcal{U}\}$. For any open set $U \subset 2^\kappa$ we define $U^* = \{\mathcal{F} \in E(2^\kappa): U \in \mathcal{F}\} = \mathrm{cl}\Pi^{-1}(U)$.

We recall the following construction of A. Dow and J. van Mill [1].

Let + denote the groupstructure on 2^κ. Fix an ultrafilter $\mathcal{U}_0 \in \Pi^{-1}(0)$. For $x \in 2^\kappa$ put $\mathcal{U}_x = x + \mathcal{U}_0 = \{x + U: U \in \mathcal{U}_0\} \in E(2^\kappa)$. Note that $\mathcal{U}_x \in \Pi^{-1}(x)$. In particular, if we put $E(\mathcal{U}_0) = \{\mathcal{U}_x: x \in 2^\kappa\} \subset E(2^\kappa)$ then $E(\mathcal{U}_0)$ is a dense subspace of $E(2^\kappa)$. Moreover, Π: $E(\mathcal{U}_0) \to 2^\kappa$ is a continuous bijection. For any subset $A \subset 2^\kappa$ put $A(\mathcal{U}_0) = \{U_a: a \in A\}$.

THEOREM 1.1: [1] If P is a P-space, then for every embedding $P \subset 2^\kappa$ and for every $\mathcal{U}_0$, the map Π: $P(\mathcal{U}_0) \to P$ is a homeomorphism. In particular, every P-space can be embedded into an extremely disconnected one.

The question for what subspaces $A \subset 2^\kappa$ the map Π: $A(\mathcal{U}_0) \to A$ is a homeomorphism I cannot answer. Assume Π: $A(\mathcal{U}_0) \to A$ is a homeomorphism. Under what condition is the map Π: $\mathrm{cl}A(\mathcal{U}_0) \to \mathrm{cl}A$ a homeomorphism? We have the following easy answer.

LEMMA 1.2: If $A \subset 2^\kappa$ such that Π: $A(\mathcal{U}_0) \to A$ is a homeomorphism, then the following are equivalent.

(1) Π: $\mathrm{cl}A(\mathcal{U}_0) \to \mathrm{cl}A$ is a homeomorphism.

(2) If U, V are pairwise-disjoint regular open sets then

$$\mathrm{cl}[\Pi(U^* \cap A(\mathcal{U}_0))] \cap \mathrm{cl}[\Pi(V^* \cap A(\mathcal{U}_0))] = \varnothing.$$

Proof: (1) $\to$ (2). U and V are disjoint open regular open subsets so $U^* \cap V$ are disjoint clopen subsets of $E(2^\kappa)$. In particular, $U^* \cap \mathrm{cl}A(\mathcal{U}_0)$ and $V^* \cap \mathrm{cl}A(\mathcal{U}_0)$ are disjoint clopen subsets of $\mathrm{cl}A(\mathcal{U}_0)$. But the map Π: $\mathrm{cl}A(\mathcal{U}_0) \to \mathrm{cl}A$ is a homeomorphism, so the sets $\Pi(U^* \cap A(\mathcal{U}_0))$ and $\Pi(V^* \cap A(\mathcal{U}_0))$ are contained in the pairwise disjoint clopen (relative to $\mathrm{cl}A$) set $\Pi(U^* \cap \mathrm{cl}A(\mathcal{U}_0))$ and $\Pi(V^* \cap \mathrm{cl}A(\mathcal{U}_0))$ and the conclusion follows.

(2) $\to$ (1). It suffices to prove that the map Π: $\mathrm{cl}A(\mathcal{U}_0) \to \mathrm{cl}A$ is injective. For $p \neq q$, choose disjoint clopen sets U^* and V^* with $p \in U^*$, $q \in V^*$. Then $p \in \mathrm{cl}(U^* \cap A(\mathcal{U}_0))$ and $q \in \mathrm{cl}(V^* \cap A(\mathcal{U}_0))$. The assumption implies that $\Pi(p) \neq \Pi(q)$. ❑

2. THE ALGEBRA $\mathcal{BC}(X)$

Let X be a compact zero-dimensional space. We introduce the following algebra.

$$\mathcal{BC}(X) = \{f^{-1}(A): A \subset 2^{\omega_0},\ f: X \to 2^{\omega_0} \text{ continuous}\}.$$

Then,

$$CO(X) \subset Bair(X) \subset \mathcal{BC}(X).$$

The following observation is easy to verify.

LEMMA 2.1: The elements of $\mathcal{BC}(X)$ are Lindelöf subsets of X and $\mathcal{BC}(X)$ is a ω_1-field of sets.

Observe that $\mathcal{BC}(X)$ need not be a $(2^{\omega_0})^+$-field of sets.

EXAMPLE: Consider the Alexander compactification αS of the Sorgenfrey-line S, the so-called "double-arrow space." then $\alpha S - S$ is homeomorphic to S, so $X = S_1 \cup S_2$ with $S_1 \approx S_2$ and $S_1 \cap S_2 = \varnothing$. Since αS is the first countable, $\{x\} \in \mathcal{BC}(\alpha S)$, for all x. The only possible supremum for the family $\{\{x\} : x \in S_1\}$ is S_1 itself, since $\alpha S - \{a\} \in \mathcal{BC}(\alpha S)$, for all $a \in S_2$. However, $S_1 \notin \mathcal{BC}(\alpha S)$, since it is easy to verify that elements $A \in \mathcal{BC}(\alpha S)$ have the property that A^2 is Lindelöf.

Note that the elements of the algebra $\mathcal{BC}(X)$ are clopen in the G_δ topology X_δ on X. Therefore, the Stone space $St(\mathcal{BC}(X))$ is a compactification of X_δ. The algebra $\mathcal{BC}(X)$ has the following property.

THEOREM 2.2: $\mathcal{BC}(X)$ is a quotient of a complete Boolean algebra.

Proof: Since $\mathcal{BC}(X)$ is a ω_1-field of sets, it follows that the Stone space $St(\mathcal{BC}(X))$ is a basically disconnected compactification of the P-space X_δ, i.e., the space X with the G_δ topology.

We construct the embedding $St(\mathcal{BC}(X))$ into some 2^κ. First we define the index set I by:

$$I = \{(f, A) \mid f : X \to 2^{\omega_0} \text{ is continuous}, A \subset 2^{\omega_0}\}.$$

Put $C = \{(f, A) : A \text{ clopen}\} \subset I$. For a subset $A \subset 2^{\omega_0}$, let $\chi_A \to \{0,1\}$ be the characteristic map. For $(f, A) \in I$, define $f_A : X \to \{0,1\}$ by $f_A = \chi_A \circ f$. Note that $(f, A) \in C \Leftrightarrow f_A$ is continuous. For any $\mathcal{G} \subset I$ define $e_{\mathcal{G}} : X \to \{0,1\}^{\mathcal{G}}$ by:

$$e_{\mathcal{G}}(x) = \{g_A(x) \mid (g, A) \in \mathcal{G}\}.$$

Note that the $e_{\mathcal{G}}$ map is continuous if $\mathcal{G} \subset C$. The following properties are easy to verify.

(1) $e_C : X \to 2^C$ is an embedding.
(2) $e_I : X_\delta \to 2^I$ is an embedding.
(3) $\mathrm{cl}(e_I(X)) \subset 2^I$ is homeomorphic to $St(\mathcal{BC}(X))$.

Consider $\mathcal{E}(2^I)$ and choose an ultrafilter $\mathcal{U}_0 \in \Pi^{-1}(0)$.

CLAIM: The embedding $e_I : e_I(X) = X_\delta \subset St(\mathcal{BC}(X)) \subset 2^I$ can be lifted to a homeomorphism of $St(\mathcal{BC}(X))$ to $\mathrm{cl}((e_I(X)(\mathcal{U}_0)) \subset \mathcal{E}(2^I)$.

We verify the conditions mentioned in Lemma 1.2. Let U be a regular open subset of $\Pi(U^* \cap e_I(X)(\mathcal{U}_0)) \subset \mathcal{E}(2^I)$. It suffices to show that $\Pi(U^* \cap e_I(X)(\mathcal{U}_0))$ has a clopen closure in $\mathrm{cl}(e_I(X))$.

There exists a countable $\mathcal{H} \subset I$ such that $U = p_H^{-1}(p_H(U))$ and $\mathrm{cl}\, U = p_H^{-1}\mathrm{cl}(p_H(U))$, where $p_H : 2^I \to 2^{\mathcal{H}}$ denotes the projection. If $\mathcal{H} = \{(h_n, H_n) : n \in \omega_0\}$ then the map $e_{\mathcal{H}} : X \to \{1,0\}^{\mathcal{H}}$ need not be continuous, but the property:

$$A \subset \{1,0\}^{\mathcal{H}} \Rightarrow e_{\mathcal{H}}^{-1}(A) \in \mathcal{BC}(X)$$

is still valid, and moreover:

$$e_{\mathcal{I}}(e_{\mathcal{H}}^{-1}(A)) \text{ has clopen closure in } \mathrm{cl}(e_{\mathcal{I}}(X)),$$

because of the following. First of all, the map $\Pi h_n\colon X \to (2^{\omega_0})^{\mathcal{H}}$ is continuous. If we define $B_A \subset (2^{\omega_0})^{\mathcal{H}}$ by

$$b = \{b_n\}_n \in B_A \Leftrightarrow \{\chi_{H_n}(b_n)\}_n \in A,$$

then it is easy to verify that

$$e_{\mathcal{H}}^{-1}(A) = (\Pi h_n)^{-1}(B_A).$$

And so the pair $(\Pi h_n, B_A)$ is represented in the index set $\mathcal{I}$, i.e., there is some $I \in \mathcal{I}$ such that the projection $p_i\colon 2^{\mathcal{I}} \to \{0,1\}$ has the property:

$$p_i^{-1}(1) \cap e_{\mathcal{I}}(X) = p_{\mathcal{H}}^{-1}(A) \cap e_{\mathcal{I}}(X).$$

But in such a situation we can conclude that $e_{\mathcal{H}}^{-1}(A)$ has clopen closure in $\mathrm{cl}\, e_{\mathcal{I}}(X)$. Finally, we apply these observations to the set:

$$A = p_{\mathcal{H}}(\Pi(U^* \cap e_{\mathcal{I}}(X)(\mathcal{U}_0))).$$

To see this, it is enough to observe that the set $B = \Pi(U^* \cap 2^{\mathcal{I}}(\mathcal{U}_0))$ also has the property that $B = p_{\mathcal{H}}^{-1}(p_{\mathcal{H}}(B))$. (This identity is verified in [1], and is the key to Theorem 1.1.) ❑

3. THE PROOF OF THEOREM 0.1

Let $\mathcal{B} \subset \mathcal{P}(D)$ be a $(2^{\omega_0})^+$-field of sets. We may assume that the elements of $\mathcal{B}$ separate the points of D. Consider the Stone space $X = St(\mathcal{B})$ of $\mathcal{B}$. Then X is $(2^{\omega_0})^+$-basically disconnnected and X contains the space D (i.e., the set D with the topology generated by $\mathcal{B}$ as an open base) as a dense subspace. This dense subspace is a P-space. Theorem 0.1 follows from Theorem 2.2 and the following claim.

CLAIM: X can be embedded in $St(\mathcal{B}C(X))$.

In fact we will show that if the algebra $\mathcal{B}C(X)$ is taken modulo the ideal of nowhere dense sets, then $\mathcal{B}$ is obtained. Let $f\colon X \to 2^{\omega_0}$ be a continuous map and choose $A \subset 2^{\omega_0}$. Because each $t \in 2^{\omega_0}$ is a closed G_δ-set and D is a P-space:

$$\mathrm{Int}(f^{-1}(t)) \neq \varnothing \Leftrightarrow f^{-1}(t) \cap D \neq \varnothing.$$

Put $B = 2^{\omega_0} - A$. Put $A_1 = f(f^{-1}(A) \cap D)$ and $B_1 = f(f^{-1}(B) \cap D)$. Then:

(1) $D \subset \bigcup\{\mathrm{Int}(f^{-1}(t)) : t \in A_1\} \cup \bigcup\{\mathrm{Int}(f^{-1}(t)) : t \in B_1\}$.

(2) The supremum of the family $\{\mathrm{Int}(f^{-1}(t)) : t \in A_1\}$ is disjoint from $\mathrm{Int}(f^{-1}(t))$, $\forall t \in B_1$.

(3) The supremum of the family $\{\mathrm{Int}(f^{-1}(t)) : t \in B_1\}$ is disjoint from $\mathrm{Int}(f^{-1}(t))$, $\forall t \in A_1$.

It follows that the set $f^{-1}(A) \in \mathcal{B}C(X)$ is equal to the clopen set: $\mathrm{Sup}\{\mathrm{Int}(f^{-1}(t)) : t \in A_1 \cup B_1\}$ except for the nowhere dense part $X - \bigcup\{\mathrm{Int}(f^{-1}(t)) : t \in A_1 \cup B_1\}$. This proof shows that the required quotient map φ: $\mathcal{B}C(X) \to \mathcal{B}$ is the map $r(C) = \mathrm{cl}(C \cap D)$. ❑

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On the Space of Homeomorphisms of a Compact *n*-manifold

RAYMOND Y. WONG[a]

Department of Mathematics
University of California
Santa Barbara, California 93106

ABSTRACT: In the study of the space of homeomorphisms $H(M)$ (with identity on the boundary) of a compact n-manifold M (modulo its boundary), the major unsolved problem is whether $H(M)$ an absolute neighborhood retract. If the answer is affirmative, then it must be homeomorphic to a Hilbert manifold when combined with known results in infinite-dimensional topology. It turns out that there is a simple reduction of the problem to the case M being an n-ball B. That is, if $H(B)$ is an absolute retract, then $H(M)$ is an absolute neighborhood retract. We intend to concentrate on the study of $H(B)$ and prove that there is dense subset of the space of homeomorphism of the Hilbert cube Q such that each of its member is a finite composition of homeomorphisms that are either Lie in $H(B)$ or act in the direction complementary to B.

1. INTRODUCTION

In this paper we fix an integer $n > 0$. Let J and J_k denote the closed interval $[-1, 1]$, I the unit interval $[0, 1]$, B the n-cube $J^n = J_1 \times J_2 \times \dots \times J_n$ and Q the Hilbert cube $J^\infty = J_1 \times J_2 \times J_3 \times \dots \times J_n$. In the study of the space of homeomorphisms $H_\partial(M)$ of a compact manifold M (having the identity on its boundary $\partial(M)$), the problem that whether $H_\partial(M)$ is an absolute neighborhood retract (ANR) reduces to whether $H_\partial(B)$ is an AR (see the remark by Haver in [5, Section HS2]). Hence we concentrate our effort to the study of $H_\partial(B)$.

We view B as the subset $B \times 0$ in Q and $H_\partial(B)$ as a subset of $H_\partial(Q)$ by identifying h with $h \times id$, where id is the identity function on $Q_n = J_{n+1} \times J_{n+2} \times \cdots$, and the boundary of Q, $\partial(Q)$, taken as $\partial(B) \times Q_n$. Since $H_\partial(Q)$ is known to be an AR [3], [8], we may ask whether there is retraction of $H_\partial(Q)$ onto $H_\partial(B)$. Of course if such a retraction exists, $H_\partial(B)$ will be an AR. The purpose of the present paper is to study such a question. In our main results, we show that there is a dense subset W of $H_\partial(Q)$ consisting of elements of the form $h_k \dots h_2 h_1$, where each $h_i \in H_\partial(Q)$ has the property that it either maps B onto B, or satisfies $ph_i = p$, with $p\colon Q \to B$ being the projection. Clearly, each h_i retracts into $H_\partial(B)$ by either the restriction on B or simply maps onto the identity of $H_\partial(B)$. It remains to be seen whether the present results can be improved to produce a uniformly continuous retraction of W onto $H_\partial(B)$.

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[a]E-mail: wong@math.ucsb.edu.

Throughout the following, we denote $s = (-1, 1)^\infty \subset Q$ and $p_i\colon Q \to I_i$ the projection onto the i-th factor of Q. We will be concerned mainly with the following stable subset of $H_\partial(Q)$:

$$\mathcal{H} = \{h \in H_\partial(Q) \mid h \text{ is the identity } (id) \text{ on a neighborhood of } \delta(Q)\}.$$

Clearly h is dense in $H_\partial(Q)$. An $h \in \mathcal{H}$ is said to be of *type* I if $h(B) = B$, *type* II if $ph = p$. Denote

$$W = \{h_k \dots h_2 h_1 \mid \text{each } h_i \in \mathcal{H} \text{ is of } type \text{ I or } type \text{ II}\}$$

and $\overline{W}$ = closure of W in $\mathcal{H}$. Our main purpose is to show that:

THEOREM A: $\overline{W} = \mathcal{H}$.

It follows that every homeomorphism on Q can be approximated by one which is the finite composition of homeomorphisms of *types* I or II.

2. PRELIMINARY RESULTS

We first state a technical result concerning a version of Z-set unknotting in Q. A subset A in a space X is called a Z-set in X if A is closed and for each open cover $\mathcal{U}$ of X, there is a map $f\colon X \to X\backslash A$ which is $\mathcal{U}$-close to id, that is, each pair of points x and $f(x)$ is contained in some member of $\mathcal{U}$.

LEMMA 1: Let A, A' be Z-sets in Q such that $(A \cup A') \cap \partial Q = \varnothing$. Suppose there is a homeomorphism $h: A \to A'$ such that $d(h, id) < \varepsilon$. Then there is $h' \in \overline{W}$ extending h such that $d(h', id) < \varepsilon$.

Proof (sketch): The difference between the stated lemma and the well-known Z-set unknotting theorem in infinite-dimensional (I-D) topology is that we require the extension to be a member in $\overline{W}$. In the following we give only a sketch of the argument. For a more detailed proof, we refer to [8].

First we connect A to A' by a homotopy g mapping $A \times I$ onto a Z-set in Q having the properties that $g_{(A \times 0)} = id$, $g_{(A \times 1)} = h$, $g_{(A \times I)} \cap \partial Q = \varnothing$ and the trace of each point $g(\{a\} \times I)$ is contained in an open convex set of mesh $< \varepsilon$. Since $g_{(A \times I)} \cap \partial Q = \varnothing$, using standard I-D topology we can map $g|_{(A \times I)}$ into s by an arbitrarily small *type* I homeomorphism in $\mathcal{H}$. So, to simplify notation, we may assume $g|_{(A \times I)} \subset s$. The standard argument in I-D topology is to extend h by a limited homeomorphism h' on Q, satisfying $d(h', id) < \varepsilon$, which is the composition of infinite many coordinate moves (a coordinate move is a $f \in W$ such that for some i, $p_k f = p_k$ for all $k \neq i$). Clearly, such a homeomorphism can be approximated by one which is the composition of finitely many coordinate moves. In other words, each component is of *type* I or II. Therefore, $h' \in W$. ❑

Our next result is a key engulfing lemma which says that for some given engulfing homeomorphism (described below) $h \in W$, any conjugation of h by a member of $\mathcal{H}$ lies in W. The homeomorphism h is simply a radial expansion on B crossed with the identity on Q_n. To describe h and the lemma, we need some notation.

If $x, y \in B$, $x \neq 0 \neq y$, let $\theta(x, y)$ represents the angle in radian between the line segments one joining 0 to x and the other joining 0 to y. Let $q: Q \to Q_n$ be the projection. For $r < t$, let B_r denote the ball of radius r in B and $B_{[r, t]}$ denote the closed annulus consisting of all point x in B such that $r < |x| < t$. Define $B_{[r, t)}$ analogously.

LEMMA 2: Given constants $0 < a < b < c < d < 1$, let h be a symmetric radial expansion on $B \times Q_{n+1}$ in the form of $h_0 \times id$, where $h_0 \in H_\partial(B)$ satisfies the following conditions.

(i) $h_0|B_a \cup (B \backslash B_d) = id$.

(ii) $\theta(x, h_0(x)) = 0$ for all $x \in B \backslash \{0\}$ and

(iii) $h_0(B_d) = B_c$.

Then for any $g \in \mathcal{H}$ and $\varepsilon > 0$, there is $\lambda \in \overline{W}$ such that

(a) $\lambda|_{g-1((B_a \cup (B \backslash B_d)) \times Q_n)} = id$,

(b) $d(\lambda, g^{-1}hg) < \varepsilon/2$ and

(c) $\lambda(g^{-1}(B_{[0, b)} \times Q_n)) \supset g^{-1}(B_c \times Q_n)$.

Proof (outline): The argument is essentially the same as those given in [6], [7]. We proceed as follow. For each $i > 0$, consider the set $A_i = g^{-1}|_{(B_{[a, d]} \times Z_i)}$, where $Z_i = \{x \in Q_n | \, x_k$ (the k-coordinate of x) $= 0$ for all $k > n + i\}$. Thus $A_1 \subset A_2 \subset \ldots$, each A_i is a Z-set in Q satisfying $A_i \cap \partial Q = \varnothing$ and $g^{-1}hg$ maps each $g^{-1}(A_i)$ onto itself.

Starting with $i = 1$, by Lemma 1 there is a $\lambda_1 \in \overline{W}$ such that

(a_1) $\lambda_1|_{g-1((B_a \cup (B \backslash B_d)) \times Q_n)} = id$,

(b_2) $d(\lambda_1, g^{-1}hg) < \varepsilon/2$ and

(c_3) $\lambda_1(g^{-1}(B_{[0, b)} \times Q_n)) \supset g^{-1}(B_c \times Z_1)$.

Inductively we can construct a sequence of homeomorphisms $\lambda_1, \lambda_2, \ldots$, each a member in $\overline{W}$, having the properties that for all i,

(a_i) $\lambda_i|_{g-1((B_a \cup (B \backslash B_d)) \times Q_n)} = id$,

(b_i) $d(\lambda_i, g^{-1}hg) < \varepsilon/2$ and

(c_i) $\lambda_i(g^{-1}(B_{[0, b)} \times Q_n)) \supset g^{-1}(B_c \times Z_i)$ and

(d_i) $\lambda_i | g^{-1}(B_{[0, b)} \times Z_{i-1}) = \lambda_i - 1$.

The key observation (as demonstrated in [8]) is that $\cup_i g^{-1}(B_c \times Z_i)$ is dense in $g^{-1}(B_c \times Q_n)$ and contained in the open set $\cup_i \lambda_i(g^{-1}(B_{[0, b)} \times Q_n))$. It follows from I-D topology that the complement

$$K = g^{-1}(B_c \times Q_n) \backslash \cup_i \lambda_i(g^{-1}(B_{[0, b)} \times Q_n)),$$

representing the part of $g^{-1}(B_c \times Q_n)$ not yet engulfed by the union $\cup_i \lambda_i(g^{-1}(B_{[0, b)} \times Q_n))$, is a (closed) Z-set in Q. Hence, as a consequence of Lemma 1 again, there is a $\lambda_\infty \in \overline{W}$ such that

$$\lambda_\infty|_{g-1((B_a \cup (B \backslash B_d)) \times Q_n)} = id,$$

$$d(\lambda_\infty, g^{-1}hg) < \varepsilon/2 \text{ and}$$

$$\lambda_\infty(g^{-1}(B_{[0,\,b)} \times Q_n)) \supset K.$$

It is the argument exhibited in [8] that, by requiring λ_∞ to satisfy some straightforward fiber preserving conditions, the composition of a finite number of λ_i together with λ_∞ suffices to produce a $\lambda = \lambda_\infty \lambda_m \ldots \lambda_2 \lambda_1$ (for sufficiently large m) so that

$$g^{-1}(B_c \times Q_n) \subset \lambda\,(g^{-1}(B_{[0,b)} \times Q_n)$$

and λ satisfies the desired properties of the lemma.

LEMMA 3. Let h be defined as in Lemma 2, then for any $g \in \mathcal{H}$, $g^{-1}hg \in \overline{W}$.

Proof: The proof employs the results of Lemma 2 repetitively to obtain engulfing homeomorphisms which, after a finite number of stages, approximate $g^{-1}hg$. The details of the argument is identical to the engulfing Theorem in Connell [2, Theorem 1] and will be omitted.

Proof of Theorem A: Let $W' = \{\lambda \in \mathcal{H} \mid g^{-1}\lambda g \in \overline{W} \text{ for every } g \in \mathcal{H}\}$. Clearly $W' \subset \overline{W}$ (letting $g = \lambda$) and is a normal subgroup of $\mathcal{H}$. But $\mathcal{H}$ is known to be a simple group [4]. By Lemma 2 W' contains an $h \neq id$. Hence, $W' = \mathcal{H}$. It follows that $\overline{W} \in \mathcal{H}$.

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Extending Invariant Measures on Topological Groups[a]

PIOTR ZAKRZEWSKI

Institute of Mathematics
University of Warsaw
ul. Banacha 2, 02097
Warsaw, Poland

ABSTRACT: Let G be an uncountable Hausdorff topological group. We prove that if G is either Polish and not locally compact or compact and not zero-dimensional, or the countable product of finite groups with uniformly bounded cardinalities, then every invariant σ-finite measure on G has a proper invariant extension. We do this by expressing G as the union of a "short" chain of its proper subgroups.

INTRODUCTION

Let G be an uncountable group.

The aim of this note is to address the question, whether every invariant σ-finite measure on G has a proper invariant extension. Harazišvili [4] and, independently, Erdös and Mauldin [2] proved that if $m: \mathcal{A} \to [0, +\infty]$ is such a measure, then its domain $\mathcal{A}$ is not equal to $\mathcal{P}(G)$ i.e., the measure m is not *universal* on G. One would like to know if there is, moreover, a *proper invariant extension* of m, i.e., a measure $m':\mathcal{A}' \to [0, +\infty]$ such that $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}' \neq \mathcal{A}$ and $m'|\mathcal{A} = m$.

All measures considered here are assumed to be σ-additive, extended real-valued, diffused and σ-finite.

We will say that a group G satisfies the *Measure Extension Property* (MEP), if every invariant measure on G has a proper invariant extension.

By an old result of Hulanicki [6], if there is no real-valued measurable cardinal $\leq |G|$, then G satisfies MEP. Pelc [9] proved that all Abelian groups satisfy MEP and conjectured that in fact *all* groups do. Though it is now known that some other algebraically defined classes of groups satisfy MEP, the question, whether Pelc's conjecture is true, remains a major open problem.

The aim of this note is to show that some topologically defined classes of groups satisfy MEP.

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RESULTS

Throughout the paper G is assummed to be a Hausdorff topological group. Our results are as follows.

THEOREM 1: If G is Polish and not locally compact, then G satisfies MEP.

THEOREM 2: If G is compact and not zero-dimensional, then G satisfies MEP.

THEOREM 3: If G is the product of a sequence $\langle G_n : n < \omega \rangle$ of finite groups with the property that there is a finite constant N such that $1 < |G_n| < N$ for each $n < \omega$, then G satisfies MEP.

In the rest of this note we outline proofs of these results.

Suppose that m is an invariant measure on G. Call a set $A \subseteq G$ *almost invariant* if $m(A \triangle gA) = 0$ for every $g \in G$.

We shall need two measure-theoretic lemmas.

LEMMA 4: (Hulanicki [6]) If there exists a nonmeasurable almost invariant set, then m has a proper invariant extension.

LEMMA 5: (Harazišvili [5]) If there is a set Z such that $m^*(Z) > 0$ and for every sequence $\langle g_k : k < \omega \rangle$ of elements of G there is a sequence $\langle f_i : i < \omega \rangle$ of elements of G such that

$$\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k Z = \varnothing,$$

then m has a proper invariant extension.

The next fact contains the key observation.

LEMMA 6: Suppose that H is a subgroup of G and the index $[G:H]$ of H in G is uncountable. If $m^*(H) > 0$, then m has a proper invariant extension.

Proof: It suffices to show that the set $Z = H$ has the property from Lemma 5.

So take an arbitrary sequence $\langle g_k : k < \omega \rangle$ of elements of G. Since the index of H in G is uncountable, it is possible to choose for each $i \in \omega$ an $f_i \in G$ such that

$$f_i \notin \bigcup_{k<\omega} g_i H g_k^{-1}. \tag{*}$$

It suffices now to check that

$$\Big(\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k H\Big) \cap \bigcup_{k<\omega} g_k H = \varnothing.$$

So suppose that there is

$$g \in \Big(\bigcap_{i<\omega} f_i \bigcup_{k<\omega} g_k H\Big) \cap \bigcup_{k<\omega} g_k H.$$

Fix $i, k \in \omega$ such that $g \in f_i g_k H \cap g_i H$. But then $f_i \in g_i H g_k^{-1}$, contradicting (*). ❑

Let us say that an infinite, regular cardinal λ is in $\mathrm{Cof}(G)$, the *cofinality spectrum of G*, if

$$G = \bigcup_{\alpha<\lambda} G_\alpha$$

for a certain strictly increasing sequence $\langle G_\alpha: \alpha < \lambda\rangle$ of subgroups of G with $[G:G_\alpha] > \omega$ for each $\alpha < \lambda$.

Note that if G is the union of a chain of length λ of proper subgroups and λ has uncountable cofinality then its cofinality $\mathrm{cf}(\lambda)$ belongs to $\mathrm{Cof}(G)$.

Recall that a universal measure on a set X is called *uniform* if all sets of cardinalities less than $|X|$ have measure zero.

The proof of the following Main Lemma is based on ideas taken from [6].

MAIN LEMMA: If there exists $\lambda \in \mathrm{Cof}(G)$ such that there is no universal, uniform measure on λ, then G satisfies MEP.

Proof: Let m be an invariant measure on G. We will show that m has a proper invariant extension.

Take $\lambda \in \mathrm{Cof}(G)$ and let $\langle G_\alpha: \alpha < \lambda\rangle$ be the corresponding sequence of subgroups.

If there is an $\alpha < \lambda$ such that $m^*(G_\alpha) > 0$, then we are done by Lemma 6.

So assume otherwise and for each $\alpha < \lambda$ let $Q_\alpha = G_{\alpha+1}\setminus G_\alpha$; by what we have just assumed, $m(Q_\alpha) = 0$. It then follows that for every $T \subseteq \lambda$, the set $A_T = \bigcup_{\alpha\in T} Q_\alpha$ is almost-invariant. Finally, for some $T \subseteq \lambda$, the set A_T is nonmeasurable, since otherwise it is easy to define a uniform universal measure on λ. By Lemma 4, this completes the proof. ❑

Let d denote the smallest size of a dominating family in ω^ω.

LEMMA 7: (Fremlin [3]) There is no universal uniform measure on $\mathrm{cf}(d)$.

Now Theorems 1, 2, and 3 will follow from Main Lemma as soon as the final three lemmas are established.

LEMMA 8: If G is Polish and not locally compact, then $\mathrm{cf}(d) \in \mathrm{Cof}(G)$.

Proof: Since G is Polish and not σ-compact, the minimal size of a collection of compact sets which covers G equals d.

Let $\{C_\alpha: \alpha < d\}$ be such a collection. In order to see that $\mathrm{cf}(d) \in \mathrm{Cof}(G)$ consider the sequence $\langle G_\alpha: \alpha < d\rangle$, where G_α is the subgroup of G generated by $\bigcup_{\xi<\alpha} C_\xi$. ❑

LEMMA 9: If G is compact and not zero-dimensional, then $\omega_1 \in \mathrm{Cof}(G)$.

Proof: The representation theory of compact groups tells us that in this case there is a (continuous) group homomorphism which maps G onto an uncountable closed subgroup G' of the group $U(n, \mathbb{C})$ of all complex $n \times n$ unitary matrices, for some $n > 0$.

It clearly suffices to prove that $\omega_1 \in \mathrm{Cof}(G')$.

The idea (which has origins in Ciesielski's paper [1]) is to represent $\mathbb{C}$ as the union of a strictly increasing sequence $\langle L_\alpha: \alpha < \omega_1\rangle$ of its subfields and then to define G_α as the subgroup of G' consisting of all matrices from G' with entries restricted to L_α. Some care must be taken, however, to secure that $G_\alpha \neq G'$. So let $\mathcal{B}$ be a maximal algebraically independent over $\mathbb{Q}$ subset of the set of all entries of the elements of G'. Represent $\mathcal{B}$ as the union of a strictly increasing sequence $\langle \mathcal{B}_\alpha: \alpha < \omega_1\rangle$ of its subsets and define L_α as the algebraic closure of the field generated by $\mathcal{B}_\alpha \cup \mathbb{Q}$. Finally, consider the sequence $\langle G_\alpha: \alpha \in \omega_1\rangle$. ❑

LEMMA 10: If G is the product of a sequence $\langle G_n: n < \omega\rangle$ of finite groups with the property that there is a finite constant N such that $1 < |G_n| < N$ for each $n < \omega$, then $\omega_1 \in \mathrm{Cof}(G)$.

Proof: By a result of van Douwen, there is a strictly increasing sequence $\langle F_\alpha: \alpha < \omega_1\rangle$ of filters in $\mathcal{P}(\omega)$ such that $U = \bigcup_{\alpha<\omega_1} F_\alpha$ is a free ultrafilter in $\mathcal{P}(\omega)$. Let:

$$H = \left\{g \in G: \{n < \omega: g_n = e_n\} \in U\right\},$$

where e_n is the neutral element of G_n.

Then H is a normal subgroup of G and $[G{:}H] < N$. Fix a selector S of the collection of all H-cosets. For each $\alpha < \omega_1$ let:

$$H_\alpha = \left\{g \in G: \{n < \omega: g_n = e_n\} \in F_\alpha\right\}$$

and define G_α as the subgroup of G generated by $H_\alpha \cup S$.

Note that $G_\alpha \neq G$, since $[G{:}\, H_\alpha] > \omega$ and the subgroup of G generated by S is at most countable. ❑

FINAL COMMENTS

The question, which cardinal numbers are in $\mathrm{Cof}(G)$ is interesting in its own right.

Sharp and Thomas [10] proved that if $\mathrm{Sym}(\omega)$ is the group of all permutations of ω, then MA implies that $\mathrm{Cof}(\mathrm{Sym}(\omega)) = \{2^\omega\}$ (compare this with Lemma 8; in fact, in [10] another proof is given that $\mathrm{cf}(d) \in \mathrm{Cof}(\mathrm{Sym}(\omega))$). On the other hand Macpherson and Neumann [8] showed that $\omega \notin \mathrm{Cof}(\mathrm{Sym}(\omega))$.

If G is compact and not zero-dimensional (see Lemma 9), then a refinement of our argument establishes that $\omega \in \mathrm{Cof}(G)$. In particular, this is true for all compact Lie groups.

If G is the countable product of finite groups with uniformly bounded cardinalities (see Lemma 10), then it may happen that $\omega \notin \mathrm{Cof}(G)$ (see [7]). On the other hand, it is easy to see that $\omega \in \mathrm{Cof}((\mathbb{Z}_2)^\omega)$.

Note finally, that if G is the countable product of a sequence of at most countable, nontrivial groups, then if all of them are either infinite or have uniformly bounded cardinalities, then G satisfies MEP. It is open what happens in the intermediate case.

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